We consider the matrix:

$$A = \begin{pmatrix} 2 & -6 & 4 \\ -2 & 1 & -2 \\ 1 & 6 & -1 \end{pmatrix}.$$

1. (4pts) Using the definitions on Slide 16,

$$\|A\|_1 = \max\{|2| + |-2| + |1|, |-6| + |1| + |6|, |4| + |-2| + |-1|\} = 13.$$

$$\|A\|_\infty = \max\{|2| + |-6| + |4|, |-2| + |1| + |-2|, |1| + |6| + |-1|\} = 12.$$

Since $\|A\|_2 = \sqrt{\lambda_{max}(A^*A)}$, we compute

$$A^*A = \begin{pmatrix} 2 & -2 & 1 \\ -6 & 1 & 6 \\ 4 & -2 & -1 \end{pmatrix} \begin{pmatrix} 2 & -6 & 4 \\ -2 & 1 & -2 \\ 1 & 6 & -1 \end{pmatrix}. = \begin{pmatrix} 9 & -8 & 11 \\ -8 & 73 & -32 \\ 11 & -32 & 21 \end{pmatrix}.$$

From MatLab, the eigenvalues are $\lambda = 0.0309, 12.9640, 90.0052$, so

$$\|A\|_2 = \sqrt{90.0052} = 9.4871.$$

the largest singular value of A.

2. (4pts) The characteristic equation for A satisfies:

$$\det \begin{vmatrix} 2-\lambda & -6 & 4 \\ -2 & 1-\lambda & -2 \\ 1 & 6 & -1-\lambda \end{vmatrix} = -(\lambda - 3)(\lambda + 2)(\lambda - 1) = 0.$$

For $\lambda_1 = 3$, we solve

$$(A - 3I)\xi_1 = \begin{pmatrix} -1 & -6 & 4 \\ -2 & -2 & -2 \\ 1 & 6 & -4 \end{pmatrix} \xi_1 = 0, \qquad \text{so} \qquad \xi_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = -2$, we solve

$$(A + 2I)\xi_2 = \begin{pmatrix} 4 & -6 & 4 \\ -2 & 3 & -2 \\ 1 & 6 & 1 \end{pmatrix} \xi_2 = 0, \qquad \text{so} \qquad \xi_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda_3 = 1$, we solve

$$(A - I)\xi_3 = \begin{pmatrix} 1 & -6 & 4 \\ -2 & 0 & -2 \\ 1 & 6 & -2 \end{pmatrix} \xi_3 = 0, \qquad \text{so} \qquad \xi_3 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}.$$

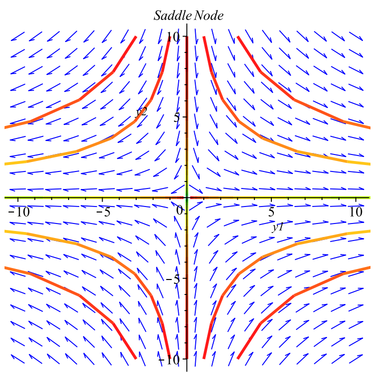
It follows that an appropriate transformation matrix is given by:

$$P = \begin{pmatrix} -2 & 1 & -2 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix} \qquad \text{and} \qquad P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

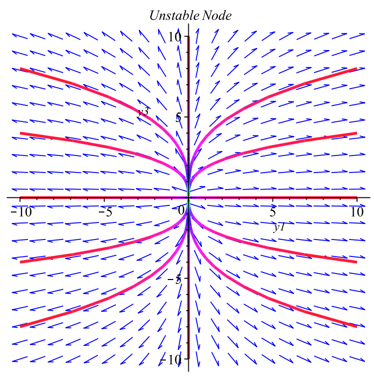
3. (4pts) From the previous problem, if we define $J = P^{-1}AP$, then the solution to the IVP $\dot{y} = Jy$ with $y(0) = y_0 = [2, -1, 1]^T$ satisfies:

$$y(t) = e^{Jt}y_0 = \begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{3t} \\ -e^{-2t} \\ e^{t} \end{pmatrix}$$

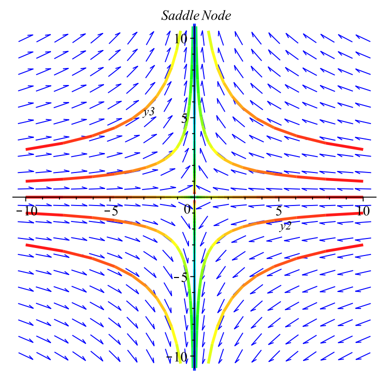
Below are the 2D phase portraits projected into the coordinate axes with the type of node labeled above the respective phase portraits.



y_1 vs y_2



y_1 vs y_3



y_2 vs y_3

4. (4pts) The solution to the original ODE, $\dot{x} = Ax$ with $x(0) = x_0$ satisfies:

$$x(t) = P \begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix} P^{-1}x(0).$$

With Maple doing the algebra we obtain:

$$x(t) = \begin{pmatrix} 2\,e^{3\,t} + e^{-2\,t} - 2\,e^{t} & 2\,e^{-2\,t} - 2\,e^{t} & 2\,e^{3\,t} - 2\,e^{t} \\ -e^{3\,t} + e^{t} & e^{t} & -e^{3\,t} + e^{t} \\ -e^{3\,t} - e^{-2\,t} + 2\,e^{t} & -2\,e^{-2\,t} + 2\,e^{t} & -e^{3\,t} + 2\,e^{t} \end{pmatrix} x_0.$$