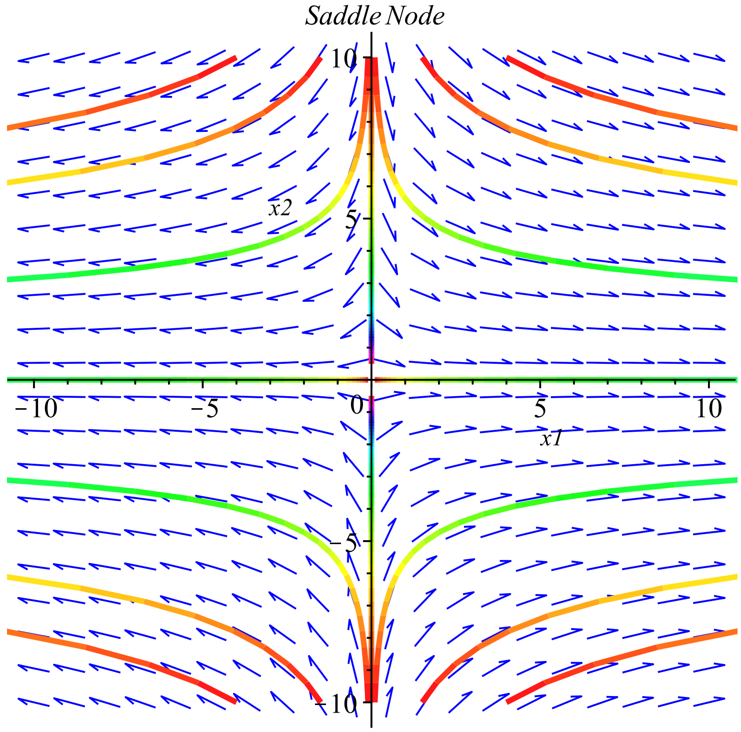
1. (5pts) The example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -0.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

has eigenvalues, $\lambda_1 = 2$ with associated eigenvector $\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda_2 = -0.5$ with associated eigenvector $\xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, since this is a diagonal matrix. It follows that the general solution is:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{-0.5t} \end{pmatrix}$$

This example produces a saddle node with the unstable direction in the x_1 direction and the stable direction following the x_2 axis. Below is a phase portrait of this system.



A fundamental solution is given by:

$$\Phi(t) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-0.5t} \end{pmatrix}.$$

2. (4pts) Prove the equivalence of $\|\cdot\|_\infty$ and $\|\cdot\|_2$, *i.e.*, show that

$$C\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le D\|\mathbf{x}\|_\infty,$$

for some constants C and D.

Proof: For some $n \geq 1$, let $\mathbf{x} = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$. Suppose that $|x_i| = M$ and $|x_j| \leq M$ for all $j \neq i$ with $1 \leq i, j \leq n$. By definition, $\|\mathbf{x}\|_\infty = M$. It follows that:

$$M^2 = \|\mathbf{x}\|_\infty^2 \leq \sum_{j=1}^{n} |x_j|^2, \qquad \text{so} \qquad \|\mathbf{x}\|_\infty \leq \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}} = \|\mathbf{x}\|_2.$$

In addition, we have

$$\sum_{j=1}^{n} |x_j|^2 \leq nM^2 = n\|\mathbf{x}\|_\infty^2, \qquad \text{so} \qquad \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}} = \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty.$$

Thus, we have shown equivalence with $C = 1$ and $D = \sqrt{n}$ and

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty.$$

3. (7pts) a. Let:

$$A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad C = A + B = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

From lecture we know $\|C\|_1$ is the maximum column sum of the absolute values of the elements, which is 5. Similarly, we know $\|C\|_\infty$ is the maximum row sum of the absolute values of the elements, which is also 5.

b. Showing that A and B commute follows from:

$$AB = 4IB = 4B = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = B(4I) = BA.$$

Since A and B commute, the lecture notes give that

$$e^{Ct} = e^{(A+B)t} = e^{At}e^{Bt}.$$

However, we have shown that

$$e^{At} = \begin{pmatrix} e^{4t} & 0 & 0 \\ 0 & e^{4t} & 0 \\ 0 & 0 & e^{4t} \end{pmatrix} = e^{4t}I.$$

From the definition of the matrix exponential, we have

$$e^{Bt} = \left(I + Bt + \frac{B^2}{2!}t^2 + ... + \frac{B^n}{n!}t^n + ...\right) = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{t^2}{2} + \mathbf{0}\right).$$

It follows that $e^{Ct} = e^{(A+B)t} = e^{At}e^{Bt} = e^{4t}e^{Bt}$. Thus, the fundamental solution is given by:

$$e^{Ct} = e^{4t}e^{Bt} = e^{4t}\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{4t} & te^{4t} & \frac{t^2e^{4t}}{2} \\ 0 & e^{4t} & te^{4t} \\ 0 & 0 & e^{4t} \end{pmatrix}.$$