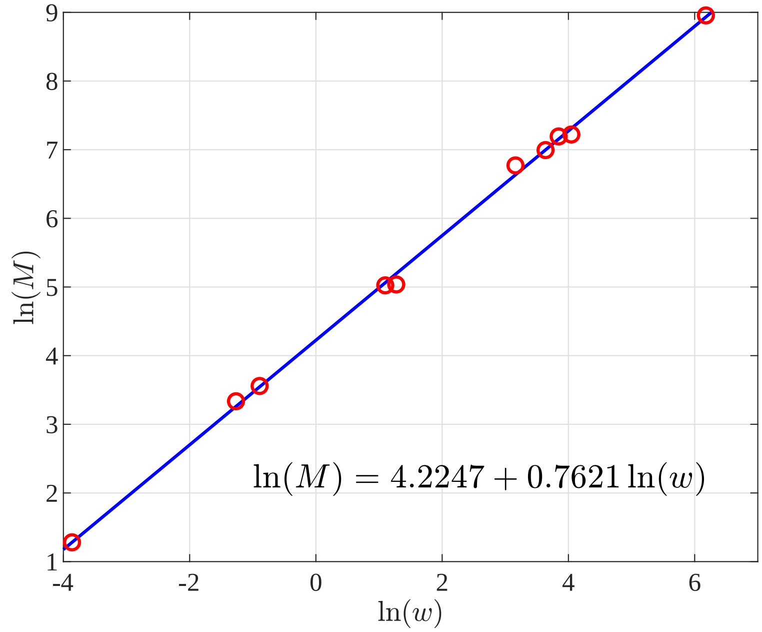
1. (6pts) The best linear fit to the logarithms of the metabolic data satisfies:

$$\ln(M) = 0.7621\ln(w) + 4.2247,$$

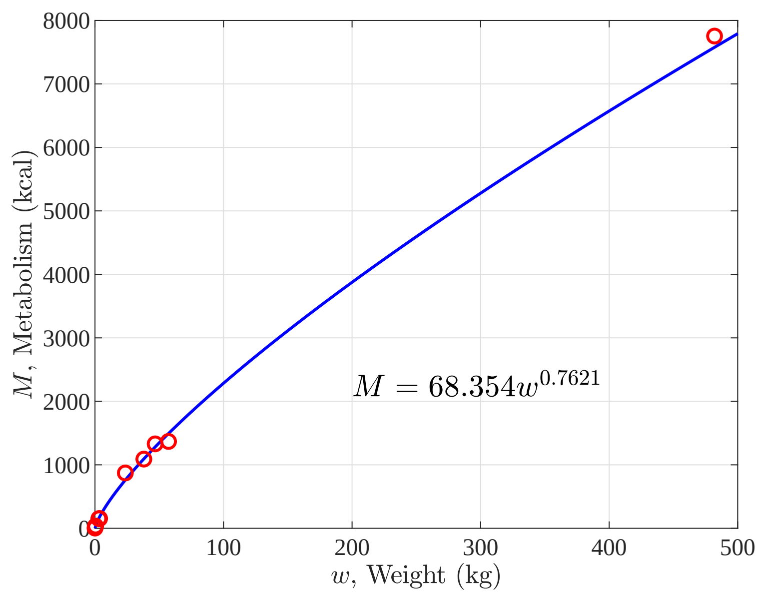
which is shown in the graph below:



This allometric model is given by:

$$M = 68.35\,w^{0.7621},$$

which is shown in the graph below:



Mammals use most of their energy generating heat, which is lost through the skin (surface area L^2). As weight is density times volume and the density is constant (mostly water), so weight is proportional to L^3. It follows that energy lost through the skin is proportional to $w^{2/3}$. However, mammals also use energy that is volumetric (proportional to w^1) through use of muscles, brain, digestion, and lungs. Thus, we expect the power in the allometric model to be greater than 2/3 (and less than 1), weighted closer to 2/3. These data give Kleiber's Law, which has $r = 0.75$ in agreement with the best fitting allometric model.

2. (5pts) The spruce budworm model is given by:

$$\frac{dB}{dt} = r_B B\left(1 - \frac{B}{K_B}\right) - \frac{\beta B^2}{\alpha^2 + B^2}.$$

We scale the population and time by $p = sB$ and $\tau = qt$, so the scaled differential equation is written:

$$\begin{aligned} \frac{dp}{d\tau} &= \frac{s}{q}\frac{dB}{dt} = \frac{s}{q}\left(r_B B\left(1 - \frac{B}{K_B}\right) - \beta\frac{B^2}{\alpha^2 + B^2}\right) \\ &= \frac{s}{q}\left(r_B\frac{p}{s}\left(1 - \frac{p}{sK_B}\right) - \beta\frac{(p/s)^2}{\alpha^2 + (p/s)^2}\right) \end{aligned}$$

Let $s = 1/\alpha$, then

$$\frac{dp}{d\tau} = \frac{r_B}{q}p\left(1 - \frac{\alpha p}{K_B}\right) - \frac{\beta}{\alpha q}\left(\frac{p^2}{1 + p^2}\right).$$

Take $q = \beta/\alpha$, then

$$\frac{dp}{d\tau} = \frac{\alpha r_B}{\beta}p\left(1 - \frac{\alpha p}{K_B}\right) - \frac{p^2}{1 + p^2}.$$

If we define the new scaled parameters,

$$R = \frac{\alpha r_B}{\beta} \qquad \text{and} \qquad Q = \frac{K_B}{\alpha},$$

then the scaled model is given by:

$$\frac{dp}{d\tau} = Rp\left(1 - \frac{p}{Q}\right) - \frac{p^2}{1 + p^2}.$$

3. (5pts) With Q = 10 and R = 0.5, the two parameter model satisfies:

$$\frac{dp}{d\tau} = 0.5p\left(1 - \frac{p}{10}\right) - \frac{p^2}{1 + p^2} = f(p).$$

The equilibria satisfy:

$$0.5p_e\left(1 - \frac{p_e}{10}\right) - \frac{p_e^2}{1 + p_e^2} = 0,$$

so clearly one equilibrium is $p_e = 0$ (extinction). Thus, it remains to solve

$$0.5\left(1 - \frac{p_e}{10}\right) = \frac{p_e}{1 + p_e^2},$$

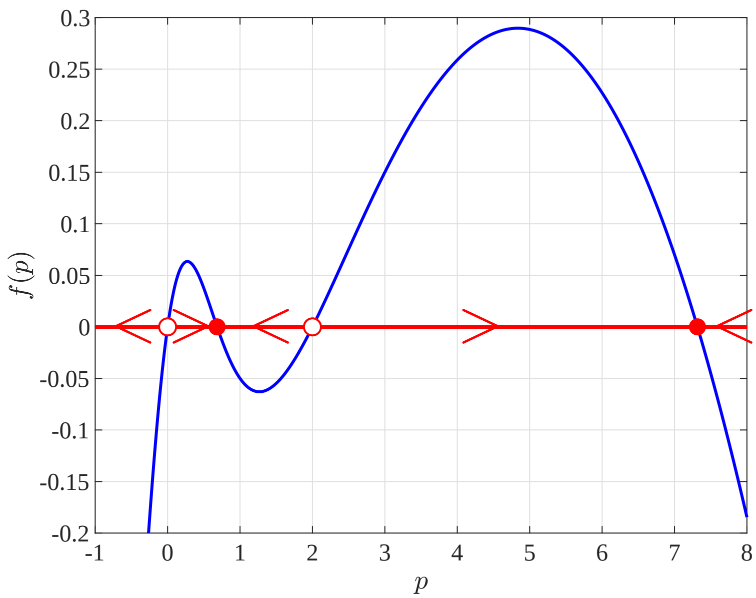
which is equivalent to solving a cubic equation (or solved with Maple). Specifically, one can show the equivalent cubic equation is

$$(p_e - 2)(p_e^2 - 8p_e + 5) = 0,$$

so the other **three** equilibria are:

$$p_e = 2, 0.68338, 7.31662.$$

To draw a 1D Phase Portrait for this model, we graph $f(p)$. The 1D Phase Portrait readily follows as seen below:



From the graph of $f(p)$ the stability of the equilibria are readily found using the direction of flow, which only depends on whether $f(p)$ is positive (arrow to the right) or negative (arrow to the left). It follows that $p_e = 0$ (extinction) is unstable, $p_e = 0.68338$ (endemic) is stable, $p_e = 2$ is unstable, and $p_e = 7.31662$ (outbreak) is stable.