This Lecture Activity has you actively work with the lecture notes presented in class and available on my website. This activity is due by **Fri. Oct 29 by noon**. The problems below require written answers, which are entered into **Gradescope**.

Note: For full credit you must show intermediate steps in your calculations.

1. (4pts) The lecture notes show how to find a second linearly independent solution when the Cauchy-Euler equation has two equal roots. This problem (and the next) examines the case when there are three equal roots, where $F(r) = (r - r_1)^3$ has the triple root r_1. Define the differential operator L by

$$L[y] = t^3 y''' + \alpha t^2 y'' + \beta t y' + \gamma y = 0.$$

Let $y(t) = t^r$ and find the corresponding *auxiliary equation*, $F(r)$ for this 3^{rd} order Cauchy-Euler equation. Assume that r_1 is a triple root, so $F(r) = (r - r_1)^3 = 0$. From the lecture notes we know that two linearly independent solutions are:

$$y_1(t) = t^{r_1} \qquad \text{and} \qquad y_2(t) = t^{r_1} \ln(t),$$

which was verified on Slide 13 by examining:

$$\frac{\partial}{\partial r} L[t^r] = \frac{\partial}{\partial r}[t^r F(r)].$$

Find the third linearly independent solution by examining:

$$\frac{\partial^2}{\partial r^2} L[t^r] = \frac{\partial^2}{\partial r^2}[t^r F(r)].$$

(Slides 13-14)

2. (4pts) Consider the 3^{rd} order linear homogeneous ODE given by:

$$t^3 y''' + 9t^2 y'' + 19ty' + 8y = 0.$$

Use similar techniques for solving the *Cauchy-Euler problem* to solve this problem. Give your *auxiliary equation* and find **3** linearly independent solutions to this problem, writing the general solution to this ODE. Problem 1 should give your third linearly independent solution. (Slides 13-14)

3. (4pts) **Reduction of Order** (Jean D'Alembert (1717-1783)): If $y_1(x)$ is known for the linear ODE:

$$y'' + p(x)y' + q(x)y = 0.$$

Then one attempts a solution of the form $y(x) = v(x)y_1(x)$. Provided $y_1(x) \neq 0$, show that

$$\frac{dv}{dx} = \frac{1}{[y_1(x)]^2} e^{-\int^x p(s)ds}.$$

Solve for $v(x)$ to obtain the 2^{nd} linearly independent solution, $y_2(x)$. (This will just be an integral expression, not easily simplified.)

4. (4pts) a. Consider the following ODE:

$$xy'' + (1-2x)y' + (x-1)y = 0. \tag{1}$$

Show that $y_1(x) = e^x$ is a solution to this differential equation.

b. In Part a, $y_1(x) = e^x$ was found as one solution to (1). Use the **Reduction of Order** method to find $y_2(x)$ for (1). Use the Wronskian to show this is a *fundamental set of solutions*.