1. (6pts) Consider the algebraic equation

$$x^2 + \varepsilon x - 1 = 0, \qquad 0 < \varepsilon \ll 1.$$

Determine the first three terms in a regular perturbation series solution

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$$

for each root. Compare your answers to the exact roots.

2. (10pts) Find a two term regular perturbation expansion for the solution of the boundary value problem (BVP):

$$\begin{aligned} y'' - \varepsilon y &= 0, \qquad && 0 < t < 1, \qquad 0 < \varepsilon \ll 1, \\ y(0) &= 0, && y(1) = 1. \end{aligned}$$

Compare this expansion to the exact solution of this BVP. Show that the two term approximate solution satisfies the BVP up to an $o(\varepsilon)$ term. How well does it satisfy the boundary conditions and does the approximation uniformly approach the solution on the interval $t \in (0,1)$?

Definition 1. *Let $f(t,\varepsilon)$ and $g(t,\varepsilon)$ be defined for all $t \in I$ and all ε in a (punctured) neighborhood of $\varepsilon = 0$. We write*

$$f(t,\varepsilon) = o\big(g(t,\varepsilon)\big), \qquad \text{as} \quad \varepsilon \to 0,$$

if

$$\lim_{\varepsilon \to 0} \left| \frac{f(t,\varepsilon)}{g(t,\varepsilon)} \right| = 0,$$

pointwise on I. If this limit is uniform on I, we write $f(t,\varepsilon) = o\big(g(t,\varepsilon)\big)$ as $\varepsilon \to 0$ uniformly on I.

3. (10pts) Consider the initial value problem (IVP):

$$\begin{aligned} \ddot{y} + (1+\varepsilon) y &= 0, \qquad && t > 0, \qquad 0 < \varepsilon \ll 1, \\ y(0) &= 1, && \dot{y}(0) = 0. \end{aligned}$$

Find the exact solution. Find a two term perturbation approximation and show that the correction term is a secular term. Compare the exact solution to the perturbation approximation for large t. Create a graph of the exact solution and the approximate solution for $t \in [0, 50]$ for $\varepsilon = 0.05$ and describe similarities and differences.

4. (12pts) Consider the BVP:

$$y'' + \varepsilon^2 y = 0, \qquad 0 < x < 1, \qquad y(0) = a, \quad y(1) = 0.$$

a. Find the regular perturbation solution to this problem up to and including terms of $\mathcal{O}\left(\varepsilon^2\right)$.

b. Obtain the exact solution for this BVP ($\varepsilon \neq n\pi$) and compare the Taylor series of this solution up to and including terms of $\mathcal{O}(\varepsilon^2)$. Why do we have the condition $\varepsilon \neq n\pi$?

c. Create graphs for the exact and approximate solutions with $a = 2$ for $\varepsilon = 0.5$ and $\varepsilon = 1$. What can you say about the differences between secular terms in a BVP as compared to an IVP with respect to convergence of the solution?

5. (12pts) Consider the IVP:

$$u'' - u = \varepsilon t u, \qquad t > 0, \qquad u(0) = 1, \quad u'(0) = -1, \qquad 0 < \varepsilon \ll 1.$$

a. Find a two term regular perturbation approximation, $u_a(t)$.

b. Use a regular power series method to solve this problem. Give your recurrence relation and show the coefficients up to and including t^6.

c. Let $\varepsilon = 0.04$ and use a numerical differential equation solver (like MatLab's ODE23) to find an accurate numerical solution for $t \in [0, 8]$. Create a graph comparing the perturbation approximation of Part a, the series solution of Part b, and the numerical solution (restricting the range to $u \leq 10$). Briefly describe what you observe and how these various methods compare.

6. (10pts) Consider the IVP:

$$y'' + y = \varepsilon y (y')^2, \qquad y(0) = 1, \quad y'(0) = 0, \qquad 0 < \varepsilon \ll 1.$$

a. Find a two term regular perturbation approximation, $y_a(t)$.

b. Let $\varepsilon = 0.1$ and use a numerical differential equation solver (like MatLab's ODE23) to find an accurate numerical solution for $t \in [0, 50]$. Create a graph comparing the perturbation approximation of Part a and the numerical solution. Briefly describe what you observe and how these solutions compare.

7. (10pts) Consider the IVP:

$$u' + u = \frac{1}{1 + \varepsilon u}, \qquad u(0) = 0, \qquad 0 < \varepsilon \ll 1.$$

a. Find a two term regular perturbation approximation, $u_a(t)$.

b. Let $\varepsilon = 0.2$ and use a numerical differential equation solver (like MatLab's ODE23) to find an accurate numerical solution for $t \in [0, 10]$. Create a graph comparing the perturbation approximation of Part a and the numerical solution. Briefly describe what you observe and how these solutions compare.

c. Start with $\varepsilon = 0.2$ and halve ε twice. Compare the approximate solution at $t = 10$, $u_a(10)$ with the accurate numerical solution, $u(10)$ for these **3** values of ε. Use this information to show that your perturbation approximation is $\mathcal{O}(\varepsilon^2)$.