The WeBWorK Power Series assignment had one problem where you write details. The details of the individual problems vary randomly.

WW Problem 5. (6pts) Consider the initial value problem:

$$(4-x^2)y'' - 2xy' + 30y = 0, \qquad y(0) = 7, \quad y'(0) = 4.$$

With a power series solution of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{so} \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

It follows that

$$(4-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + 30\sum_{n=0}^{\infty} a_n x^n = 0,$$

or

$$4\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + 30\sum_{n=0}^{\infty} a_n x^n = 0.$$

The 1^{st} term can have indices shifted, while the 2^{nd} and 3^{rd} terms can start at $n = 0$. Thus, we have:

$$4\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}\Big(n(n-1) + 2n - 30\Big)a_n x^n = 0.$$

From this equation, we obtain the recurrence relation as follows:

$$a_{n+2} = \frac{n^2+n-30}{4(n+2)(n+1)}a_n = \frac{(n-5)(n+6)}{4(n+2)(n+1)}a_n, \qquad n = 0, 1, 2, ...$$

The initial conditions give $a_0 = 7$ and $a_1 = 4$. From the recurrence relation, we obtain the following:

$$\begin{aligned}
a_2 &= \frac{(-5)(6)}{4(2)(1)}a_0 = \frac{-30}{8}(7) = -\frac{105}{4}, \\
a_3 &= \frac{(-4)(7)}{4(3)(2)}a_1 = \frac{-28}{24}(4) = -\frac{14}{3}, \\
a_4 &= \frac{(-3)(8)}{4(4)(3)}a_2 = \frac{-24}{48}\left(\frac{-105}{4}\right) = \frac{105}{8}, \\
a_5 &= \frac{(-2)(9)}{4(5)(4)}a_3 = \frac{-18}{80}\left(\frac{-14}{3}\right) = \frac{21}{20}, \\
a_6 &= \frac{(-1)(10)}{4(6)(5)}a_4 = \frac{-10}{120}\left(\frac{105}{8}\right) = -\frac{35}{32}, \\
a_7 &= \frac{(0)(11)}{4(7)(6)}a_5 = 0.
\end{aligned}$$

It follows that

$$y_1(x) = a_0\left(1 - \frac{15}{4}x^2 + \frac{15}{8}x^4 - \frac{5}{32}x^6 + \dots\right),$$

while

$$y_2(x) = a_1 \left(x - \frac{7}{6}x^3 + \frac{21}{80}x^5 \right).$$

Thus, $y_2(x)$ is a polynomial, so converges for all x, while $y_1(x)$ is an infinite series. We apply the ratio test to successive terms in y_1, giving:

$$\lim_{n\to\infty} \frac{|a_{n+2}x^{n+2}|}{|a_n x^n|} = \lim_{n\to\infty} \left| \frac{(n-5)(n+6)}{4(n+2)(n+1)} \right| x^2 = \frac{x^2}{4} < 1.$$

Thus, this series converges absolutely for $|x| < 2$, which is the radius where the ODE becomes singular.