1. (8pts) Consider the initial-value problem

$$y'' + p(x)y' + q(x)y = 0, \qquad y(0) = y_0, \quad y'(0) = v_0.$$

We assume a transformation of the form $y = vY$, so

$$y' = vY' + v'Y \qquad \text{and} \qquad y'' = vY'' + 2v'Y' + v''Y,$$

which is substituted into the original equation. This gives

$$\begin{aligned} vY'' + 2v'Y' + v''Y + p(x)\big(vY' + v'Y\big) + q(x)vY &= 0, \\ Y'' + \left(\frac{2v'}{v} + p\right)Y' + \left(\frac{v''}{v} + \frac{pv'}{v} + q\right)Y &= 0. \end{aligned}$$

To eliminate the Y' term, we must have:

$$\frac{2v'}{v} + p(x) = 0 \qquad \text{or} \qquad \frac{v'}{v} = -\tfrac{1}{2}p(x).$$

Integrating both sides and exponentiating gives:

$$v(x) = e^{-\frac{1}{2}\int_0^x p(s)ds}.$$

It follows that

$$v'(x) = -\frac{p(x)}{2}e^{-\frac{1}{2}\int_0^x p(s)ds} \qquad \text{and} \qquad v''(x) = \left(-\frac{p'(x)}{2} + \left(\frac{p(x)}{2}\right)^2\right)e^{-\frac{1}{2}\int_0^x p(s)ds}.$$

To obtain the form $Y'' + Q(x)Y = 0$, we need:

$$\begin{aligned} Q(x) &= \frac{v''}{v} + p(x)\frac{v'}{v} + q(x), \\ Q(x) &= -\frac{p'(x)}{2} + \left(\frac{p(x)}{2}\right)^2 + p(x)\left(-\frac{p(x)}{2}\right) + q(x), \\ Q(x) &= -\frac{p'(x)}{2} - \frac{p^2(x)}{4} + q(x). \end{aligned}$$

From above we see

$$Y(0) = \frac{y(0)}{v(0)} = y_0, \qquad \text{since} \qquad v(0) = e^0 = 1,$$

and

$$y'(0) = v'(0)Y(0) + v(0)Y'(0) = -\frac{p(0)y_0}{2} + Y'(0), \qquad \text{since} \qquad v'(0) = -\frac{p(0)}{2}.$$

It follows that $Y'(0) = v_0 + \frac{p(0)y_0}{2}$.

2. a. (10pts) Consider the singular second order ODE given by:

$$2x^2y'' + xy' + x^2y = 0.$$

With $P(x) = 2x^2$, $Q(x) = x$, and $R(x) = x^2$, we see that

$$\lim_{x\to 0} \frac{xQ(x)}{P(x)} = \lim_{x\to 0} \frac{x^2}{2x^2} = \tfrac{1}{2} = p_0,$$

and

$$\lim_{x\to 0} \frac{x^2R(x)}{P(x)} = \lim_{x\to 0} \frac{x^4}{2x^2} = 0 = q_0.$$

Since these are both finite, it follows that the functions are analytic, which implies that $x = 0$ is a regular singular point. Thus, we may attempt solutions of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}, \qquad y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}.$$

This also gives the indicial equation:

$$r(r-1) + p_0r + q_0 = r^2 - r + \tfrac{1}{2}r = r\left(r - \tfrac{1}{2}\right) = 0.$$

It follows that $r = 0, \frac{1}{2}$.

We substitute our power series into the ODE and obtain:

$$2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0,$$

which shifting indices gives:

$$2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0.$$

When $n = 0$, we also obtain the indicial equation, $a_0\big(2r(r-1) + r\big) = a_0\big(r(2r-1)\big) = 0$. Note that when $n = 1$, we have $a_1\big(2r(r+1) + r + 1\big) = a_1\big(2r^2 + 3r + 1\big) = 0$, which implies $a_1 = 0$ for solutions of the indicial equation. For $n \geq 2$, we obtain the recurrence relation:

$$a_n = -\frac{a_{n-2}}{(n+r)(2n+2r-1)}.$$

The first solution satisfies $r_1 = \frac{1}{2}$ with a_0 arbitrary and the recurrence relation, $a_n = -\frac{a_{n-2}}{n(2n+1)}$, so

$$y_1(x) = \sqrt{x}\sum_{n=0}^{\infty} a_n x^n,$$

where

$$a_2 = -\tfrac{a_0}{2\cdot 5}, \quad a_4 = -\tfrac{a_2}{4\cdot 9} = \tfrac{a_0}{2\cdot 5\cdot 4\cdot 9}, \quad a_6 = -\tfrac{a_4}{6\cdot 13} = -\tfrac{a_0}{2\cdot 5\cdot 4\cdot 9\cdot 6\cdot 13} \quad \cdots$$

The second solution satisfies $r_2 = 0$ with b_0 arbitrary and the recurrence relation, $b_n = -\frac{b_{n-2}}{n(2n-1)}$, so

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n,$$

where

$$b_2 = -\frac{a_0}{2\cdot 3}, \quad b_4 = -\frac{b_2}{4\cdot 7} = \frac{b_0}{2\cdot 3\cdot 4\cdot 7}, \quad b_6 = -\frac{b_4}{6\cdot 11} = -\frac{b_0}{2\cdot 3\cdot 4\cdot 7\cdot 6\cdot 11} \quad \cdots$$

The complete solution satisfies:

$$\begin{aligned} y(x) &= a_0\sqrt{x}\left(1 - \tfrac{1}{10}x^2 + \tfrac{1}{360}x^4 - \tfrac{1}{28080}x^6 + \tfrac{1}{3818880}x^8 + \mathcal{O}\left(x^{10}\right)\right) \\ &\quad + b_0\left(1 - \tfrac{1}{6}x^2 + \tfrac{1}{168}x^4 - \tfrac{1}{11088}x^6 + \tfrac{1}{1330560}x^8 + \mathcal{O}\left(x^{10}\right)\right). \end{aligned}$$

b. (10pts) Consider the singular second order ODE given by:

$$x^2y'' + 3xy' + (1+x)y = 0.$$

With $P(x) = x^2$, $Q(x) = 3x$, and $R(x) = 1 + x$, we see that

$$\lim_{x\to 0}\frac{xQ(x)}{P(x)} = \lim_{x\to 0}\frac{3x^2}{x^2} = 3 = p_0,$$

and

$$\lim_{x\to 0}\frac{x^2R(x)}{P(x)} = \lim_{x\to 0}\frac{x^2+x^3}{x^2} = 1 = q_0.$$

Since these are both finite, it follows that the functions are analytic, which implies that $x = 0$ is a regular singular point. Thus, we may attempt solutions of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}, \qquad y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}.$$

This also gives the indicial equation:

$$r(r-1) + p_0 r + q_0 = r^2 - r + 3r + 1 = (r+1)^2 = 0.$$

It follows that $r = -1$ is a double root.

We substitute our power series into the ODE and obtain:

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + 3\sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0,$$

which shifting indices gives:

$$\sum_{n=0}^{\infty} a_n\Big((n+r)(n+r-1) + 3(n+r) + 1\Big)x^{n+r} + \sum_{n=1}^{\infty} a_{n-1}x^{n+r} = 0.$$

When $n = 0$, we also obtain the indicial equation, $a_0(r+1)^2 = 0$. For $n \geq 1$, we obtain the recurrence relation:

$$a_n = -\frac{a_{n-1}}{(n+r)(n+r+2)+1} = -\frac{a_{n-1}}{(n+r+1)^2}, \qquad n = 1, 2, ...$$

The first solution satisfies $r_1 = -1$ with a_0 arbitrary and the recurrence relation, $a_n = -\frac{a_{n-1}}{n^2}$, so

$$y_1(x) = \frac{1}{x}\sum_{n=0}^{\infty} a_n x^n,$$

where

$$a_1 = -\frac{a_0}{1}, \quad a_2 = -\frac{a_1}{2^2} = \frac{a_0}{(2!)^2}, \quad a_3 = -\frac{a_2}{3^2} = -\frac{a_0}{(3!)^2}, \ldots, a_n = (-1)^n \frac{a_0}{(n!)^2}, \ldots$$

Thus, the first solution is given by:

$$y_1(x) = \frac{a_0}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} x^n = \frac{a_0}{x} \left(1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + ... \right).$$

Since $r = -1$ is a repeated root, if we take $y_1(x)$ above with $a_0 = 1$, then the second solution has the form:

$$y_2(x) = y_1(x) \ln(x) + x^r \sum_{n=1}^{\infty} b_n(r) x^n,$$

where $b_n(r) = a_n'(r)$ and

$$a_n'(-1) = \frac{d}{dr} \left[\frac{(-1)^n}{((n+r+1)!)^2} \right] \Bigg|_{r=-1}.$$

From the lecture notes, we saw that if $f(x) = (x - \alpha_1)^{\beta_1} \cdot \cdots \cdot (x - \alpha_n)^{\beta_n}$, then:

$$\frac{f'(x)}{f(x)} = \frac{\beta_1}{x - \alpha_1} + ... + \frac{\beta_n}{x - \alpha_n}, \quad \text{for} \quad x \neq \alpha_1, \alpha_2, ..., \alpha_n.$$

Therefore,

$$\begin{aligned} a_n'(-1) &= \left[\left(\frac{-2}{r+2} + \frac{-2}{r+3} + ... + \frac{-2}{n+r+1} \right) \cdot \left(\frac{(-1)^n}{((n+r+1)!)^2} \right) \right] \Bigg|_{r=-1} \\ &= \left(-2 \sum_{m=1}^{n} \frac{1}{m} \right) \left(\frac{(-1)^n}{(n!)^2} \right) = -2 \frac{(-1)^n}{(n!)^2} \cdot H_n, \end{aligned}$$

where $H_n = \sum_{m=1}^{n} \frac{1}{m} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n}$. It follows that:

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) - \frac{2}{x} \sum_{n=1}^{\infty} \frac{(-1)^n H_n x^n}{(n!)^2} \\ &= y_1(x) \ln(x) - \frac{2}{x} \left[-x + \frac{x^2}{(2!)^2} \left(1 + \frac{1}{2} \right) - \frac{x^3}{(3!)^2} \left(1 + \frac{1}{2} + \frac{1}{3} \right) + \frac{x^4}{(4!)^2} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) + ... \right] \end{aligned}$$

Alternately, one could take the form of $y_2(x)$ and insert that into the original ODE. The result is:

$$\sum_{k=2}^{\infty} k(k-1) b_{k+1} x^k + \sum_{k=1}^{\infty} 3k b_{k+1} x^k + \sum_{k=0}^{\infty} b_{k+1} x^k + \sum_{k=1}^{\infty} b_k x^k = -2x y_1' - 2y_1.$$

Carefully matching the same powers of x gives the same coefficients b_k listed above and below.

Combining these results give:

$$\begin{aligned} y(x) &= a_0 \tfrac{1}{x} \left(1 - x + \tfrac{1}{4} x^2 - \tfrac{1}{36} x^3 + \tfrac{1}{576} x^4 - \tfrac{1}{14400} x^5 + \tfrac{1}{518400} x^6 + \mathcal{O}\left(x^7\right) \right) \\ &\quad + b_0 \left(\tfrac{\ln(x)}{x} \left(1 - x + \tfrac{1}{4} x^2 - \tfrac{1}{36} x^3 + \tfrac{1}{576} x^4 - \tfrac{1}{14400} x^5 + \tfrac{1}{518400} x^6 + \mathcal{O}\left(x^7\right) \right) \right. \\ &\quad \left. + \tfrac{1}{x} \left(2\,x - \tfrac{3}{4} x^2 + \tfrac{11}{108} x^3 - \tfrac{25}{3456} x^4 + \tfrac{137}{432000} x^5 - \tfrac{49}{5184000} x^6 + \mathcal{O}\left(x^7\right) \right) \right). \end{aligned}$$

c. (10pts) Consider the singular second order ODE given by:

$$x^2y'' + 4xy' + (2+x)y = 0.$$

With $P(x) = x^2$, $Q(x) = 4x$, and $R(x) = 2+x$, we see that

$$\lim_{x\to 0} \frac{xQ(x)}{P(x)} = \lim_{x\to 0} \frac{4x^2}{x^2} = 4 = p_0,$$

and

$$\lim_{x\to 0} \frac{x^2R(x)}{P(x)} = \lim_{x\to 0} \frac{2x^2+x^3}{x^2} = 2 = q_0.$$

Since these are both finite, it follows that the functions are analytic, which implies that $x = 0$ is a regular singular point. Thus, we may attempt solutions of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad y' = \sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1}, \qquad y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}.$$

This also gives the indicial equation:

$$r(r-1) + p_0 r + q_0 = r^2 - r + 4r + 2 = (r+1)(r+2) = 0.$$

It follows that $r_1 = -1$ and $r_2 = -2$.

We substitute our power series into the ODE and obtain:

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + 4\sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + 2\sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0,$$

which shifting indices gives:

$$\sum_{n=0}^{\infty} a_n\Big((n+r)(n+r-1) + 4(n+r) + 2\Big)x^{n+r} + \sum_{n=1}^{\infty} a_{n-1}x^{n+r} = 0.$$

When $n = 0$, we also obtain the indicial equation, $a_0(r^2+3r+2) = a_0(r+1)(r+2) = 0$. For $n \geq 1$, we obtain the recurrence relation:

$$a_n = -\frac{a_{n-1}}{(n+r)(n+r+3)+2} = -\frac{a_{n-1}}{(n+r+1)(n+r+2)}, \qquad n = 1, 2, ...$$

The first solution satisfies $r_1 = -1$ with a_0 arbitrary and the recurrence relation, $a_n = -\frac{a_{n-1}}{n(n+1)}$, so

$$y_1(x) = \frac{1}{x}\sum_{n=0}^{\infty} a_n x^n,$$

where

$$a_1 = -\tfrac{a_0}{2}, \quad a_2 = -\tfrac{a_1}{2\cdot 3} = \tfrac{a_0}{2!3!}, \quad a_3 = -\tfrac{a_2}{3\cdot 4} = -\tfrac{a_0}{3!4!}, \ldots, a_n = (-1)^n \tfrac{a_0}{n!(n+1)!}, \ldots$$

Thus, the first solution is given by:

$$y_1(x) = \frac{a_0}{x}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!((n+1)!}x^n = \frac{a_0}{x}\left(1 - \frac{x}{2!} + \frac{x^2}{2!3!} - \frac{x^3}{3!4!} + ...\right).$$

Since $r_2 = -2$ and $r_1 - r_2 = 1$ is an integer, we evaluate:

$$\lim_{r \to r_2} a_N(r) = \lim_{r \to -2} a_1(r) = \frac{-a_0(r)}{(r+2)(r+3)}.$$

Since a_0 is an arbitrary, the limit is undefined, so a second series solution requires the logarithmic term. We take $y_1(x)$ above with $a_0 = 1$, then the second solution has the form:

$$y_2(x) = y_1(x)\ln(x) + x^{-2} \sum_{n=0}^{\infty} b_n(r) x^n.$$

This is readily substituted into the original ODE giving:

$$\begin{aligned} 2kxy_1' - ky_1 + \sum_{n=0}^{\infty}(n-2)(n-3)b_n x^{n-2} + 4ky_1 + \sum_{n=0}^{\infty} 4(n-2)b_n x^{n-2} + \\ \sum_{n=0}^{\infty} 2b_n x^{n-2} + \sum_{n=0}^{\infty} b_n x^{n-1} &= 0, \end{aligned}$$

which is readily transformed into the equation:

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)b_n x^{n-2} + \sum_{n=1}^{\infty} b_{n-1} x^{n-2} &= -k(3y_1 + 2xy_1') \\ &= -k\left(\frac{3}{x} - \frac{3}{2} + \frac{x}{4} - \frac{x^2}{48} + \cdots - \frac{2}{x} + \frac{x}{6} - \frac{x^2}{36} + \dots\right). \end{aligned}$$

When $n = 1$, we have $b_0 x^{-1} = -kx^{-1}$ or $k = -b_0$, where b_0 is arbitrary. The series produced from b_1 reproduces the solution $y_1(x)$, so we take $b_1 = 0$. The next few coefficients are readily found:

$$\begin{aligned} n = 2: &\qquad 2b_2 + b_1 = -\frac{3b_0}{2} \quad \text{or} \quad b_2 = -\frac{3b_0}{4}, \\ n = 3: &\qquad 6b_3 + b_2 = \frac{b_0}{4} + \frac{b_0}{6}, \quad \text{or} \quad b_3 = \frac{7b_0}{36}, \\ n = 4: &\qquad 12b_4 + b_3 = -\frac{3b_0}{144} - \frac{b_0}{36}, \quad \text{or} \quad b_4 = -\frac{35b_0}{1728}. \end{aligned}$$

Alternately, from the lecture notes, we compute k from:

$$k = \lim_{r \to -2}(r+2)a_1(r) = \lim_{r \to -2} \frac{(r+2)(-1)}{(r+2)(r+3)} = -1,$$

and calculate $b_n(r_2)$ from:

$$b_n(-2) = \frac{d}{dr}\left[(r+2)a_n(r)\right]_{r=-2}.$$

Following the techniques similar to those in 2(b) are used to derive the coefficients b_n. Combining these results give:

$$\begin{aligned} y(x) = \; & a_0 \tfrac{1}{x}\left(1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{144}x^3 + \tfrac{1}{2880}x^4 - \tfrac{1}{86400}x^5 + \tfrac{1}{3628800}x^6 + \mathcal{O}\left(x^7\right)\right) \\ & + b_0\Big(-\tfrac{\ln(x)}{x}\left(1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{144}x^3 + \tfrac{1}{2880}x^4 - \tfrac{1}{86400}x^5 + \tfrac{1}{3628800}x^6 + \mathcal{O}\left(x^7\right)\right) \\ & + \tfrac{1}{x^2}\left(1 - \tfrac{3}{4}x^2 + \tfrac{7}{36}x^3 - \tfrac{35}{1728}x^4 + \tfrac{101}{86400}x^5 - \tfrac{7}{162000}x^6 + \mathcal{O}\left(x^7\right)\right)\Big). \end{aligned}$$

3. a. (10pts) Bessel's equation of order $\frac{1}{2}$ satisfies:

$$x^2y'' + xy' + \left(x^2 - \tfrac{1}{4}\right)y = 0,$$

and is important in solving PDEs with spherical geometry. With $P(x) = x^2$, $Q(x) = x$, and $R(x) = x^2 - \frac{1}{4}$, we see that

$$\lim_{x\to 0}\frac{xQ(x)}{P(x)} = \lim_{x\to 0}\frac{x^2}{x^2} = 1 = p_0, \quad \text{and} \quad \lim_{x\to 0}\frac{x^2R(x)}{P(x)} = \lim_{x\to 0}\frac{x^4 - \frac{1}{4}x^2}{x^2} = -\frac{1}{4} = q_0.$$

Since these are both finite, it follows that the functions are analytic, which implies that $x = 0$ is a regular singular point. Thus, we may attempt solutions of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}, \qquad y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}.$$

These are substituted into Bessel's equation, giving:

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \tfrac{1}{4}\sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

This becomes:

$$\sum_{n=0}^{\infty}\left((n+r)(n+r-1) + (n+r) - \tfrac{1}{4}\right)a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r} = 0,$$

or

$$\sum_{n=0}^{\infty}\left((n+r)^2 - \tfrac{1}{4}\right)a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r} = 0,$$

For $n = 0$ and $n = 1$, we see

$$\left(r^2 - \tfrac{1}{4}\right)a_0 = 0 \qquad \text{and} \qquad \left((r+1)^2 - \tfrac{1}{4}\right)a_1 = 0.$$

The first gives the *indicial equation* and is satisfied by $r = \pm\frac{1}{2}$. With either value of r, the second equation implies that $a_1 = 0$. For $n \geq 2$, we have:

$$\left((n+r)^2 - \tfrac{1}{4}\right)a_n + a_{n-2} = 0 \qquad \text{or} \qquad a_n = -\frac{a_{n-2}}{(n+r)^2 - \frac{1}{4}},$$

which is the *recurrence relation.*

The first root, $r_1 = \frac{1}{2}$, is inserted into the recurrence relation to give:

$$a_n = -\frac{a_{n-2}}{\left(n+\frac{1}{2}\right)^2 - \frac{1}{4}} = -\frac{a_{n-2}}{n(n+1)}, \qquad n \geq 2.$$

Since $a_1 = 0$, it follows that $a_3 = a_5 = \cdots = a_{2m+1} = 0, \quad m \geq 0$. Continuing we see that:

$$a_2 = -\frac{a_0}{2\cdot 3} = -\frac{a_0}{3!}, \qquad a_4 = -\frac{a_2}{4\cdot 5} = \frac{a_0}{5!}, \qquad \ldots \qquad a_{2m} = (-1)^m\frac{a_0}{(2m+1)!}, \qquad m \geq 1.$$

It follows that the first solution to this Bessel's equation is:

$$y_1(x) = a_0 x^{\frac{1}{2}} \left(1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(2m+1)!} \right) = a_0 x^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+1}}{(2m+1)!} = a_0 x^{-\frac{1}{2}} \sin(x).$$

Since $r_1 - r_2 = 1$, we investigate:

$$\lim_{r \to r_2} a_N(r) = \lim_{r \to -\frac{1}{2}} a_1(r) = 0.$$

This limit exists, so the logarithmic form of $y_2(x)$ is unnecessary. It follows that the recurrence relation for r_2 satisfies:

$$b_n(r_2) = -\frac{b_{n-2}}{\left(n - \frac{1}{2}\right)^2 - \frac{1}{4}} = -\frac{b_{n-2}}{n(n-1)}, \qquad n \geq 2.$$

We note that b_0 is arbitrary and b_1 generates the same series as $y_1(x)$, so take $b_1 = 0$. Thus, $b_3 = b_5 = \cdots = b_{2m+1} = 0, \quad m \geq 0$. It follows that:

$$b_2 = -\frac{b_0}{2!}, \qquad b_4 = -\frac{b_2}{3 \cdot 4} = \frac{b_0}{4!}, \qquad \ldots \quad b_{2m} = (-1)^m \frac{b_0}{(2m)!}, \qquad m \geq 1.$$

The second linearly independent solution is:

$$y_2(x) = b_0 x^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = b_0 x^{-\frac{1}{2}} \cos(x).$$

The general solution to Bessel's equation of order $\frac{1}{2}$ is:

$$y(x) = x^{-\frac{1}{2}} \left(a_0 \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+1}}{(2m+1)!} + b_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \right) = x^{-\frac{1}{2}} \Big(a_0 \sin(x) + b_0 \cos(x) \Big).$$

b. (7pts) Consider the change of variables, $y(x) = x^{-\frac{1}{2}} v(x)$. It follows that

$$y'(x) = x^{-\frac{1}{2}} v'(x) - \tfrac{1}{2} x^{-\frac{3}{2}} v(x),$$

and

$$y''(x) = x^{-\frac{1}{2}} v''(x) - x^{-\frac{3}{2}} v'(x) + \tfrac{3}{4} x^{-\frac{5}{2}} v(x).$$

Substituting this into Bessel's equation gives:

$$x^{\frac{3}{2}} v'' - x^{\frac{1}{2}} v' + \tfrac{3}{4} x^{-\frac{1}{2}} v + x^{\frac{1}{2}} v' - \tfrac{1}{2} x^{-\frac{1}{2}} v + x^{\frac{3}{2}} v - \tfrac{1}{4} x^{-\frac{1}{2}} v = 0,$$

which reduces to

$$x^{\frac{3}{2}} \Big(v'' + v \Big) = 0 \qquad \text{or} \qquad v'' + v = 0.$$

The characteristic equation for this equation in v is $\lambda^2 + 1 = 0$, so $\lambda = \pm i$, giving the general solution:

$$v(x) = c_1 \cos(x) + c_2 \sin(x) \qquad \text{or} \qquad y(x) = x^{-\frac{1}{2}} \Big(c_1 \cos(x) + c_2 \sin(x) \Big),$$

which are the same solutions formulated by the Method of Frobenius. Note that Bessel's equation of order $\frac{1}{2}$ have solutions:

$$J_{-\frac{1}{2}}(x) = \left(\tfrac{2}{\pi x}\right)^{\frac{1}{2}} \cos(x) \qquad \text{and} \qquad J_{\frac{1}{2}}(x) = \left(\tfrac{2}{\pi x}\right)^{\frac{1}{2}} \sin(x),$$

which are appropriately scaled functions of $y(x)$.

4. a. (10pts) Consider the singular second order ODE given by:

$$x^2 y'' + 6xy' + \left(6 - x^2\right) y = 0. \tag{1}$$

With $P(x) = x^2$, $Q(x) = 6x$, and $R(x) = 6 - x^2$, we see that

$$\lim_{x\to 0} \frac{xQ(x)}{P(x)} = \lim_{x\to 0} \frac{6x^2}{x^2} = 6 = p_0, \quad \text{and} \quad \lim_{x\to 0} \frac{x^2 R(x)}{P(x)} = \lim_{x\to 0} \frac{6x^2 - x^4}{x^2} = 6 = q_0.$$

Since these are both finite, it follows that the functions are analytic, which implies that $x = 0$ is a regular singular point. Thus, we may attempt solutions of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \qquad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}.$$

These are substituted into (1), giving:

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + 6 \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + 6 \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0.$$

This becomes:

$$\sum_{n=0}^{\infty} (n+r+2)(n+r+3) a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0.$$

For $n = 0$ and $n = 1$, we see

$$(r+2)(r+3)a_0 = 0 \qquad \text{and} \qquad (r+3)(r+4)a_1 = 0.$$

The first gives the *indicial equation* and is satisfied by $r_1 = -2$ and $r_2 = -3$. With $r_1 = -2$, the second equation implies that $a_1(-2) = 0$. For $n \geq 2$, we have:

$$a_n(r) = \frac{a_{n-2}(r)}{(n+r+2)(n+r+3)},$$

which is the *recurrence relation.*

The first root, $r_1 = -2$, is inserted into the recurrence relation to give:

$$a_n = \frac{a_{n-2}}{n(n+1)}, \qquad n \geq 2.$$

Since $a_1 = 0$, it follows that $a_3 = a_5 = \cdots = a_{2m+1} = 0, \quad m \geq 0$. Continuing we see that:

$$a_2 = \frac{a_0}{2 \cdot 3} = \frac{a_0}{3!}, \qquad a_4 = \frac{a_2}{4 \cdot 5} = \frac{a_0}{5!}, \quad \ldots \quad a_{2m} = \frac{a_0}{(2m+1)!}, \quad m \geq 1.$$

It follows that the first solution to (1) is:

$$y_1(x) = a_0 x^{-2}\left(\sum_{m=0}^{\infty} \frac{x^{2m}}{(2m+1)!}\right) = a_0 x^{-3} \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!} = a_0 x^{-3}\sinh(x).$$

Since $r_1 - r_2 = 1$, we investigate:

$$\lim_{r \to r_2} a_N(r) = \lim_{r \to -3} a_1(r) = 0.$$

This limit exists, so the logarithmic form of $y_2(x)$ is unnecessary. It follows that the recurrence relation for r_2 satisfies:

$$b_n(r_2) = \frac{b_{n-2}}{(n-1)n}, \qquad n \geq 2.$$

We note that b_0 is arbitrary and b_1 generates the same series as $y_1(x)$, so take $b_1 = 0$. Thus, $b_3 = b_5 = \cdots = b_{2m+1} = 0, \quad m \geq 0$. It follows that:

$$b_2 = \frac{b_0}{2!}, \qquad b_4 = \frac{b_2}{3 \cdot 4} = \frac{b_0}{4!}, \quad \ldots \quad b_{2m} = \frac{b_0}{(2m)!}, \quad m \geq 1.$$

The second linearly independent solution is:

$$y_2(x) = b_0 x^{-3} \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} = b_0 x^{-3}\cosh(x).$$

The general solution to (1) is:

$$y(x) = x^{-3}\left(a_0 \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!} + b_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}\right) = x^{-3}\Big(a_0 \sinh(x) + b_0 \cosh(x)\Big).$$

b. (7pts) Consider the change of variables, $y(x) = x^{\alpha} v(x)$. It follows that

$$y'(x) = \alpha x^{\alpha-1} v(x) + x^{\alpha} v'(x),$$

and

$$y''(x) = \alpha(\alpha-1)x^{\alpha-2}v(x) + 2\alpha x^{\alpha-1}v'(x) + x^{\alpha}v''(x).$$

Substituting this into (1) gives:

$$\alpha(\alpha-1)x^{\alpha}v + 2\alpha x^{\alpha+1}v' + x^{\alpha+2}v'' + 6\alpha x^{\alpha}v + 6x^{\alpha+1}v' + 6x^{\alpha}v - x^{\alpha+2}v = 0,$$

or

$$x^{\alpha+2}v'' + x^{\alpha+1}\big(2\alpha+6\big)v' + \big[\alpha(\alpha-1)x^{\alpha} + 6\alpha x^{\alpha} + 6x^{\alpha} - x^{\alpha+2}\big]v = 0.$$

We choose α such that $2\alpha + 6 = 0$ or $\alpha = -3$ to eliminate the v' term. It follows that:

$$x^{-1}v'' + \big[12x^{-3} - 18x^{-3} + 6x^{-3} - x^{-1}\big]v = 0,$$

or

$$x^{-1}v'' - x^{-1}v = 0, \qquad \text{so} \qquad v'' - v = 0.$$

This ODE in $v(x)$ has the characteristic equation $\lambda = \pm 1$, so has the general solution:

$$v(x) = c_1 e^{x} + c_2 e^{-x} = d_1 \cosh(x) + d_2 \sinh(x),$$

using a different linear combination of the exponentials, where $c_1 = \frac{d_1+d_2}{2}$ and $c_2 = \frac{d_1-d_2}{2}$. Since $y(x) = x^{\alpha} v(x)$, it follows that:

$$y(x) = x^{-3}\Big(d_1 \cosh(x) + d_2 \sinh(x)\Big),$$

which are the same solutions formulated by the Method of Frobenius.

5. a. (10pts) Consider the singular second order ODE given by:

$$xy'' + (1+2x)y' + (x+1)y = 0. \tag{2}$$

With $P(x) = x$, $Q(x) = 1-2x$, and $R(x) = x-1$, we see that

$$\lim_{x\to 0} \frac{xQ(x)}{P(x)} = \lim_{x\to 0} \frac{x+2x^2}{x} = 1 = p_0, \quad \text{and} \quad \lim_{x\to 0} \frac{x^2R(x)}{P(x)} = \lim_{x\to 0} \frac{x^3+x^2}{x} = 0 = q_0.$$

Since these are both finite, it follows that the functions are analytic, which implies that $x = 0$ is a regular singular point. Thus, we may attempt solutions of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \qquad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}.$$

These are substituted into (2), giving:

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Shifting indices to match powers of x, this becomes:

$$\sum_{n=0}^{\infty} a_n (n+r)^2 x^{n+r-1} + 2\sum_{n=1}^{\infty} a_{n-1}(n+r-1) x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0.$$

For $n = 0$, we have $a_0 r^2 = 0$, which gives the indicial equation, $r^2 = 0$, so $r_1 = r_2 = 0$. For $n = 1$, we have $a_1(r+1)^2 + (2r+1)a_0 = 0$. For $n \geq 2$, we have

$$\sum_{n=2}^{\infty} \Big(a_n (n+r)^2 + (2n+2r-1)a_{n-1} + a_{n-2}\Big) x^{n+r-1} = 0,$$

which gives the recurrence relation:

$$a_n(r) = -\frac{(2n+2r-1)a_{n-1}(r) + a_{n-2}(r)}{(n+r)^2}, \qquad n \geq 2.$$

The first root, $r_1 = 0$, gives $a_1 = -a_0$ and is inserted into the recurrence relation to give:

$$a_n = -\frac{(2n-1)a_{n-1} + a_{n-2}}{n^2}, \qquad n \geq 2.$$

It follows that

$$\begin{aligned}
a_2 &= -\frac{3a_1 + a_0}{2^2} = \frac{3a_0 - a_0}{2^2} = \frac{a_0}{2} = \frac{a_0}{2!}, \\
a_3 &= -\frac{5a_2 + a_1}{3^2} = -\frac{\frac{5a_0}{2} - a_0}{3^2} = -\frac{3a_0}{2 \cdot 3^2} = -\frac{a_0}{3!}, \\
a_4 &= -\frac{7a_3 + a_2}{4^2} = \frac{\frac{7a_0}{3!} - \frac{a_0}{2!}}{4^2} = \frac{4a_0}{3!4^2} = \frac{a_0}{4!}, \\
&\vdots \\
a_n &= (-1)^n \frac{a_0}{n!}.
\end{aligned}$$

Thus, the first solution to (2) is:

$$y_1(x) = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = a_0 e^{-x}.$$

Since $r_1 = r_2 = 0$, the second solution has the form:

$$y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^n,$$

so

$$y'_2 = y'_1 \ln(x) + \frac{y_1}{x} + \sum_{n=1}^{\infty} n b_n x^{n-1}, \qquad y''_2 = y''_1 \ln(x) + 2\frac{y'_1}{x} - \frac{y_1}{x^2} + \sum_{n=2}^{\infty} n(n-1) b_n x^{n-2}.$$

Substituting this into (2) gives:

$$\begin{aligned}
x\left(y''_1 \ln(x) + 2\frac{y'_1}{x} - \frac{y_1}{x^2} + \sum_{n=2}^{\infty} n(n-1) b_n x^{n-2}\right) &+ \\
(1+2x)\left(y'_1 \ln(x) + \frac{y_1}{x} + \sum_{n=1}^{\infty} n b_n x^{n-1}\right) &+ \\
(x+1)\left(y_1 \ln x + \sum_{n=1}^{\infty} b_n x^n\right) &= 0.
\end{aligned}$$

This simplifies to:

$$\sum_{n=2}^{\infty} n(n-1) b_n x^{n-1} + (1+2x) \sum_{n=1}^{\infty} n b_n x^{n-1} + (x+1) \sum_{n=1}^{\infty} b_n x^n = -2y_1 - 2y'_1 = 0,$$

since $y_1(x) = a_0 e^{-x} = -y'_1(x)$. Shifting indices so that all terms have x^n, the above equation becomes:

$$\sum_{n=1}^{\infty} (n+1) n b_{n+1} x^n + 2\sum_{n=1}^{\infty} n b_n x^n + \sum_{n=0}^{\infty} (n+1) b_{n+1} x^n + \sum_{n=1}^{\infty} b_n x^n + \sum_{n=2}^{\infty} b_{n-1} x^n = 0.$$

For $n = 0$, we see that $b_1 = 0$. For $n = 1$, it follows that $4b_2 + 3b_1 = 0$ or $b_2 = 0$. For $n \geq 2$, we have the recurrence relation:

$$(n+1)^2 b_{n+1} + (2n+1)b_n + b_{n-1} = 0 \qquad \text{or} \qquad b_{n+1} = -\frac{(2n+1)b_n + b_{n-1}}{(n+1)^2}.$$

Thus, $b_3 = 0, \ldots, b_n = 0$, for all n. Hence, $y_2(x) = y_1(x)\ln(x) = a_0 e^{-x}\ln(x)$, so the general solution is:

$$y(x) = a_0 e^{-x} + b_0 e^{-x}\ln(x).$$

b. (6pt) Consider the linear ODE, $y'' + p(x)y' + q(x)y = 0$. If $y_1(x)$ is one solution, then one attempts a solution of the form $y(x) = v(x)y_1(x)$ to find the second solution. We saw that $v(x)$ satisfies:

$$v(x) = \int \frac{e^{-\int p(x)\cdot dx}}{[y_1(x)]^2} dx.$$

Since $y_1(x) = e^{-x}$ is one solution to (2), we have:

$$y_2(x) = e^{-x}\int \frac{e^{-\int(\frac{1}{x}+2)dx}}{e^{-2x}} dx = e^{-x}\int \frac{x^{-1}e^{-2x}}{e^{-2x}} dx = e^{-x}\ln|x|.$$

Alternately, let $y = e^{-x}v$. so $y' = e^{-x}(v' - v)$ and $y'' = e^{-x}(v'' - 2v' + v)$. When this is inserted into (2), we have:

$$xe^{-x}(v'' - 2v' + v) + (1+2x)e^{-x}(v' - v) + (1+x)e^{-x}v = 0,$$

or

$$e^{-x}(xv'' + v') = 0.$$

Let $w = v'$, then

$$w' = -\frac{w}{x} \quad \text{or} \quad \ln(w) = -\ln(x) + c \quad \text{or} \quad w(x) = v'(x) = \frac{c_1}{x}.$$

Integrating this gives:

$$v(x) = c_1\ln(x) + c_2, \qquad \text{so} \qquad y(x) = (c_1\ln(x) + c_2)e^{-x}.$$

The solutions match with the solutions from Part a.

6. (10pt) Consider the singular second order ODE given by:

$$xy'' - (2 + x^2)y' + xy = 0. \tag{3}$$

With $P(x) = x$, $Q(x) = -2 - x^2$, and $R(x) = x$, we see that

$$\lim_{x\to 0}\frac{xQ(x)}{P(x)} = \lim_{x\to 0}\frac{-2x - x^3}{x} = -2 = p_0, \quad \text{and} \quad \lim_{x\to 0}\frac{x^2R(x)}{P(x)} = \lim_{x\to 0}\frac{x^3}{x} = 0 = q_0.$$

Since these are both finite, it follows that the functions are analytic, which implies that $x = 0$ is a regular singular point. Thus, we may attempt solutions of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}, \qquad y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}.$$

These are substituted into (3), giving:

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} - 2\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$$
$$-\sum_{n=0}^{\infty} a_n(n+r)x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0,$$

so matching terms and shifting indices gives:

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-3)x^{n+r-1} - \sum_{n=2}^{\infty} a_{n-2}(n+r-3)x^{n+r-1} = 0.$$

It follows that for $n=0$, $a_0 r(r-3) = 0$, which give the indicial equation with roots, $r_1 = 3$ and $r_2 = 0$. For $n = 1$, we have $a_1(r+1)(r-2) = 0$, which implies $a_1 = 0$ for r_1. The recurrence relation becomes:

$$a_n(r) = \frac{a_{n-2}(r)}{n+r}, \qquad n \geq 2.$$

With the first root, $r_1 = 3$, we see that all odd coefficients are $a_1 = a_3 = \cdots = a_{2k+1} = 0$, and the recurrence relation is written:

$$a_n = \frac{a_{n-2}}{n+3}, \qquad n \geq 2.$$

It follows that

$$\begin{aligned}
a_2 &= \frac{a_0}{5}, \\
a_4 &= \frac{a_2}{7} = \frac{a_0}{5\cdot 7}, \\
a_6 &= \frac{a_4}{9} = \frac{a_0}{5\cdot 7\cdot 9}, \\
&\vdots \qquad \vdots \\
a_{2k} &= \frac{a_0}{5\cdot 7\cdot \cdots \cdot (2k+3)}.
\end{aligned}$$

Thus, the first solution to (3) is:

$$y_1(x) = a_0 x^3 \left(1 + \frac{x^2}{5} + \frac{x^4}{35} + \frac{x^6}{315} + ...\right).$$

Since $r_1 - r_2 = 3$, the second solution has the form:

$$y_2(x) = k y_1(x)\ln(x) + x^0 \sum_{n=0}^{\infty} b_n x^n,$$

so

$$y'_2 = k\left(y'_1 \ln(x) + \frac{y_1}{x}\right) + \sum_{n=1}^{\infty} n b_n x^{n-1}, \qquad y''_2 = k\left(y''_1 \ln(x) + 2\frac{y'_1}{x} - \frac{y_1}{x^2}\right) + \sum_{n=2}^{\infty} n(n-1) b_n x^{n-2}.$$

Substituting this into (3) gives:

$$x\left(k\left(y_1''\ln(x)+2\frac{y_1'}{x}-\frac{y_1}{x^2}\right)+\sum_{n=2}^{\infty}n(n-1)b_nx^{n-2}\right)-$$
$$(2+x^2)\left(k\left(y_1'\ln(x)+\frac{y_1}{x}\right)+\sum_{n=1}^{\infty}nb_nx^{n-1}\right)+$$
$$x\left(ky_1(x)\ln(x)+\sum_{n=0}^{\infty}b_nx^n\right) = 0.$$

This simplifies to:

$$-2b_1+\sum_{n=1}^{\infty}\Big[(n+1)(n-2)b_{n+1}-(n-2)b_{n-1}\Big]x^n = k\left(\frac{3}{x}+x\right)y_1-2ky_1'.$$

Expanding the left hand side, we see the x^2 term is zero and we have:

$$-2b_1+(-2b_2+b_0)x+(4b_4-b_2)x^3+(15b_5-2b_3)x^4+(18b_6-3b_4)x^5+\dots$$

In $y_1(x)$, we let $a_0=1$, then expanding the right hand side gives:

$$k\left(\frac{3}{x}+x\right)\left(x^3+\frac{x^5}{5}+\frac{x^7}{35}+\frac{x^9}{315}+...\right)-2k\left(3x^2+x^4+\frac{x^6}{5}+\frac{x^8}{35}+...\right)$$
$$=k\left(-3x^2-\frac{2}{5}x^4-\frac{4}{35}x^6-\frac{6}{315}x^8-...\right).$$

The leading term on the rhs is $-3kx^2$, which must be zero as there are no x^2 terms on the lhs. It follows that $k=0$. Alternately, with $r_1-r_2=3$, we examine:

$$k=\lim_{r\to r_2}(r-r_2)a_3(r)=0,$$

as $a_3=0$. The leading term on the lhs must satisfy $b_1=0$. This simplifies our series expression to:

$$\sum_{n=1}^{\infty}\Big[(n+1)(n-2)b_{n+1}-(n-2)b_{n-1}\Big]x^n=0,$$

which has the recurrence relation:

$$b_{n+1}=\frac{b_{n-1}}{n+1}.$$

With b_0 arbitrary and $b_1=b_3=b_5=\cdots=b_{2k+1}=0$, we obtain the even coefficients:

$$\begin{aligned}
b_2 &= \frac{b_0}{2} = \frac{b_0}{1!2^1},\\
b_4 &= \frac{b_2}{4} = \frac{b_0}{2!\cdot 2^2} = ,\\
b_6 &= \frac{b_4}{6} = \frac{b_0}{3!\cdot 2^3},\\
&\vdots \qquad \vdots\\
b_{2k} &= \frac{b_0}{k!\cdot 2^k}.
\end{aligned}$$

Thus, the second solution is

$$y_2(x) = b_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{2^m \cdot m!} = b_0 \sum_{m=0}^{\infty} \frac{(\frac{x^2}{2})^m}{m!} = b_0 e^{\frac{x^2}{2}}.$$

b. (6pt) To verify that $y_1(x) = e^{\frac{x^2}{2}}$ is one solution, we compute

$$y'_1(x) = xe^{\frac{x^2}{2}} \qquad \text{and} \qquad y''_1(x) = e^{\frac{x^2}{2}} + x^2 e^{\frac{x^2}{2}} = (1+x^2)e^{\frac{x^2}{2}}.$$

Substituting back into the ODE:

$$x(1+x^2)e^{\frac{x^2}{2}} - (2+x^2)xe^{\frac{x^2}{2}} + xe^{\frac{x^2}{2}} = 0,$$

which is clearly satisfied, so $y_1(x) = e^{\frac{x^2}{2}}$ is a solution to (3). This matches the result of $y_2(x)$ from Part a.

With $p(x) = -\frac{(2+x^2)}{x} = -\frac{2}{x} - x$ and using the Reduction of Order method to obtain the second solution, we obtain:

$$\begin{aligned} y_2(x) &= y_1(x) \int \frac{e^{-\int p(x)dx}}{(y_1(x))^2} dx, \\ &= e^{\frac{x^2}{2}} \int \frac{e^{\int (\frac{2}{x}+x)}}{e^{x^2}} dx, \\ &= e^{\frac{x^2}{2}} \int \frac{x^2 e^{\frac{x^2}{2}}}{e^{x^2}} dx. \end{aligned}$$

It follows that the second solution is:

$$y_2(x) = e^{\frac{x^2}{2}} \int x^2 e^{-\frac{x^2}{2}} dx.$$