Fall 2021 Homework 3 – Fundamental Solutions – Solutions Math 537

Below are the written problems from WeBWorK.

1. (5pts) The system of differential equations given by:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 9 & -8 \\ 6 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

has the characteristic equation:

$$\det \begin{vmatrix} 9-\lambda & -8 \\ 6 & -5-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1) = 0,$$

which has the eigenvalues and associated eigenvectors:

$$\lambda_1 = 3, \quad \xi^{(1)} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = 1, \quad \xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This gives the general solution:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}.$$

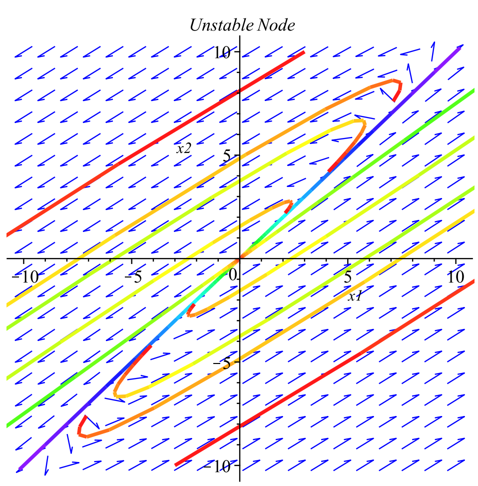
This solution has two positive eigenvalues, so the result is an unstable node or source. The transformation matrix, P, comes from the eigenvectors, giving the Jordan canonical form, J:

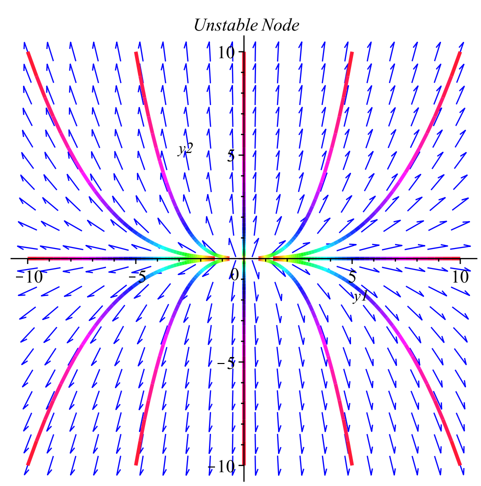
$$P = \begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} -3 & 4 \\ 1 & -1 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Thus, for $\mathbf{x} = P\mathbf{y}$, we have $\dot{\mathbf{y}} = J\mathbf{y}$, which has the solution:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}.$$

From the largest eigenvalue as $t \to \infty$, the solution moves parallel to the direction: $\xi^{(1)} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ or the y_2-axis in the transformed coordinate system. Below are the phase portraits created in Maple for the x_1x_2-coordinate system and the transformed y_1y_2-coordinate system:





2. (5pts) The system of differential equations given by:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 14 & 16 \\ -12 & -14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

has the characteristic equation:

$$\det \begin{vmatrix} 14-\lambda & 16 \\ -12 & -14-\lambda \end{vmatrix} = \lambda^2 - 4 = (\lambda-2)(\lambda+2) = 0,$$

which has the eigenvalues and associated eigenvectors:

$$\lambda_1 = 2, \quad \xi^{(1)} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -2, \quad \xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives the general solution:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 4 \\ -3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}.$$

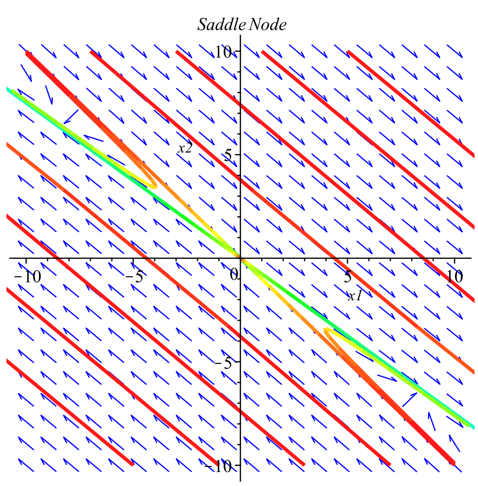
This solution has a positive and a negative eigenvalue, so the result is an saddle node. The transformation matrix, P, comes from the eigenvectors, giving the Jordan canonical form, J:

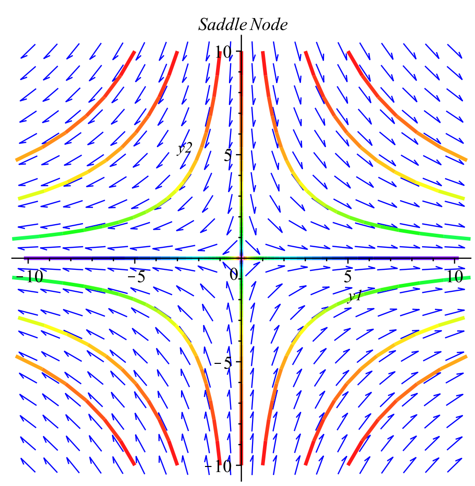
$$P = \begin{pmatrix} 4 & 1 \\ -3 & -1 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} 1 & 1 \\ -3 & -4 \end{pmatrix}, \qquad J = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Thus, for $\mathbf{x} = P\mathbf{y}$, we have $\dot{\mathbf{y}} = J\mathbf{y}$, which has the solution:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}.$$

As $t \to \infty$, the solution approaches: $\xi^{(1)} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ or the y_1-axis in the transformed coordinate system, while as $t \to -\infty$, the solution approaches: $\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or the y_2-axis in the transformed coordinate system. Below are the phase portraits created in Maple for the x_1x_2-coordinate system and the transformed y_1y_2-coordinate system:





3. (5pts) The system of differential equations given by:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & -5 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

has the characteristic equation:

$$\det \begin{vmatrix} -\lambda & -5 \\ 5 & -\lambda \end{vmatrix} = \lambda^2 + 25 = 0, \qquad \text{so} \quad \lambda = \pm 5i.$$

which has the eigenvalue and associated eigenvector:

$$\lambda_1 = 5i, \quad \xi^{(1)} = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

It follows that:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} (\cos(5t) + i\sin(5t)) = \begin{pmatrix} -\sin(5t) \\ \cos(5t) \end{pmatrix} + i \begin{pmatrix} \cos(5t) \\ \sin(5t) \end{pmatrix}.$$

This gives the general (real) solution:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} -\sin(5t) \\ \cos(5t) \end{pmatrix} + c_2 \begin{pmatrix} \cos(5t) \\ \sin(5t) \end{pmatrix}.$$

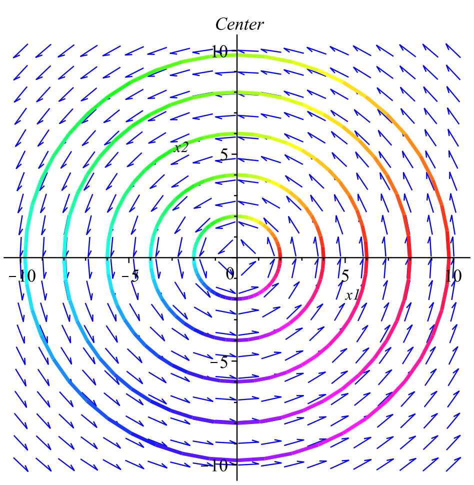
This solution has two imaginary eigenvalues, so the result is a center. All solutions form circles moving counterclockwise around the origin. The transformation matrix, P, comes from the real and imaaginary parts of the eigenvector, giving the real Jordan canonical form, J:

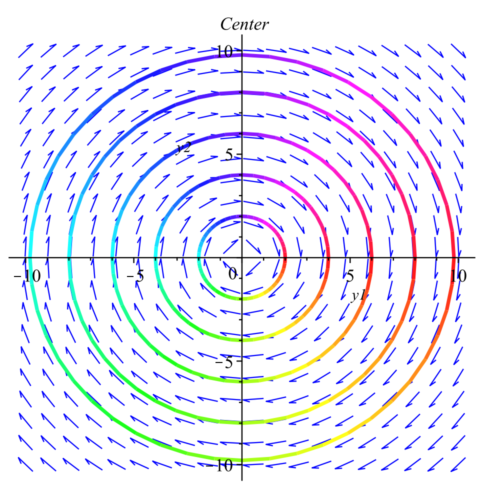
$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix}.$$

Thus, for $\mathbf{x} = P\mathbf{y}$, we have $\dot{\mathbf{y}} = J\mathbf{y}$, which has the solution:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \cos(5t) & -\sin(5t) \\ \sin(5t) & \cos(5t) \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}.$$

The transformed coordinate system changes the solutions to forming circles moving clockwise around the origin. Below are the phase portraits created in Maple for the x_1x_2-coordinate system and the transformed y_1y_2-coordinate system:





4. (5pts) The system of differential equations given by:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

has the characteristic equation:

$$\det \begin{vmatrix} 3-\lambda & -4 \\ 1 & 3-\lambda \end{vmatrix} = (\lambda - 3)^2 + 4 = 0, \qquad \text{so} \quad \lambda = 3 \pm 2i.$$

which has the eigenvalue and associated eigenvector:

$$\lambda_1 = 3 + 2i, \quad \xi^{(1)} = \begin{pmatrix} 2i \\ 1 \end{pmatrix}.$$

It follows that:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 2i \\ 1 \end{pmatrix} e^{3t}(\cos(2t) + i\sin(2t)) = e^{3t}\left(\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + i \begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix} \right).$$

This gives the general (real) solution:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}.$$

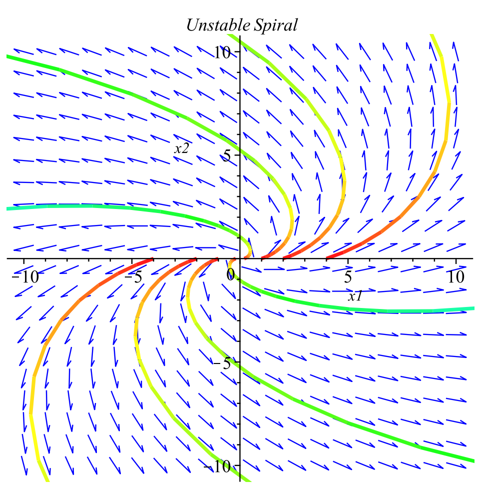
This solution has two complex eigenvalues with positive real part, so the result is an unstable spiral. All solutions form outward spirals moving counterclockwise away from the origin. The transformation matrix, P, comes from the real and imaaginary parts of the eigenvector, giving the real Jordan canonical form, J:

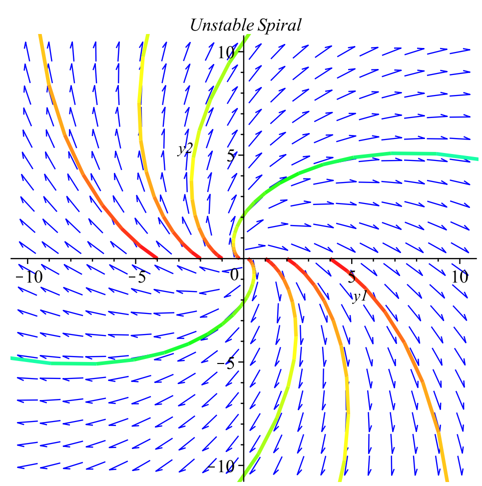
$$P = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} 0 & 0.5 \\ 1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}.$$

Thus, for $\mathbf{x} = P\mathbf{y}$, we have $\dot{\mathbf{y}} = J\mathbf{y}$, which has the solution:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{3t}\cos(2t) & -e^{3t}\sin(2t) \\ e^{3t}\sin(2t) & e^{3t}\cos(2t) \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}.$$

The transformed coordinate system changes the solutions to forming clockwise unstable spirals. Below are the phase portraits created in Maple for the x_1x_2-coordinate system and the transformed y_1y_2-coordinate system:





1. (6pts) a. Define the matrices B and C:

$$B = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that these matrices commute, $BC = CB$, so it follows that:

$$e^{At} = e^{Bt+Ct} = e^{Bt} \cdot e^{Ct}.$$

By the definition, we have

$$e^{Bt} = I + Bt + \frac{B^2t^2}{2!} + \cdots = \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix} = e^{-2t}I.$$

and

$$e^{Ct} = I + Ct + \frac{C^2t^2}{2!} + \frac{C^3t^3}{3!} + \ldots$$

This expansion becomes:

$$e^{Ct} = I + t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \tfrac{t^2}{2!} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \tfrac{t^3}{3!} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \ldots$$

Thus,

$$e^{At} = e^{Bt} \cdot e^{Ct} = e^{-2t}I \left(I + t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \tfrac{t^2}{2!} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} e^{-2t} & te^{-2t} & \frac{t^2}{2!}e^{-2t} \\ 0 & e^{-2t} & te^{-2t} \\ 0 & 0 & e^{-2t} \end{pmatrix},$$

which is the fundamental matrix solution obtained from the notes on the Jordan canonical form.

b. With A given by:

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \ldots & \ldots & 0 & 0 \end{pmatrix},$$

it is easy to see that A times any matrix B results in the rows of B shifting up one row and the last row replaced by a row of zeros. It follows that A^k results in the super-diagonal of ones in A shifting up $k-1$ rows for $k = 1, 2, \ldots, n-1$. (See Part a.) We now apply the definition of e^{At} and obtain:

$$e^{At} = I + At + A^2\tfrac{t^2}{2!} + \cdots + A^{n-1}\tfrac{t^{n-1}}{(n-1)!}.$$

or

$$e^{At} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \ldots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \ddots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & t \\ 0 & \ldots & \ldots & 0 & 1 \end{pmatrix}.$$

2. (3pts) Assuming that P is nonsingular, we first note that $(P^{-1}AP)^2 = P^{-1}AP \cdot P^{-1}AP = P^{-1}A^2P$. By induction we assume $(P^{-1}AP)^k = P^{-1}A^kP$, then

$$(P^{-1}AP)^{k+1} = P^{-1}A^kP \cdot P^{-1}AP = P^{-1}A^{k+1}P.$$

From the matrix definition of e^A, we have

$$\begin{aligned}
P^{-1}e^AP &= P^{-1}\left(I + tA + \tfrac{t^2}{2!}A^2 + \cdots + \tfrac{t^k}{k!}A^k + \dots\right)P \\
&= P^{-1}IP + tP^{-1}AP + \tfrac{t^2}{2!}P^{-1}A^2P + \cdots + \tfrac{t^k}{k!}P^{-1}A^kP + \dots \\
&= I + t(P^{-1}AP) + \tfrac{t^2}{2!}(P^{-1}AP)^2 + \cdots + \tfrac{t^k}{k!}(P^{-1}AP)^k + \dots \\
&= e^{P^{-1}AP}.
\end{aligned}$$

3. (4pts) Assume that $AB = BA$ (commute). We have the following:

$$\begin{aligned}
e^{A+B} &= \sum_{m=0}^{\infty} \tfrac{1}{m!}(A+B)^m, \\
&= \sum_{m=0}^{\infty} \tfrac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} A^kB^{m-k}, \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{m} \tfrac{1}{k!(m-k)!}A^kB^{m-k}, \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{m} \left(\tfrac{1}{k!}A^k\right)\left(\tfrac{1}{(m-k)!}B^{m-k}\right), \\
&= \sum_{k=0}^{\infty}\sum_{m=k}^{\infty} \left(\tfrac{1}{k!}A^k\right)\left(\tfrac{1}{(m-k)!}B^{m-k}\right), \\
&= \left(\sum_{k=0}^{\infty} \tfrac{1}{k!}A^k\right)\left(\sum_{j=0}^{\infty} \tfrac{1}{j!}B^j\right) \;=\; e^A \cdot e^B \;=\; e^B \cdot e^A.
\end{aligned}$$

With the commutativity of A and B we used the binomial theorem in the second line, and the order didn't matter allowing the conclusion that

$$e^{A+B} = e^Ae^B = e^Be^A.$$

4. (4pts) Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We see that these matrices fail to commute as

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By definition with $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

$$e^A = I + A + A^2 + \cdots = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Similarly,

$$e^B = I + B + B^2 + \cdots = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Thus, we have

$$e^A e^B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad e^B e^A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

We note that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{2k} = I \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{2k+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so by the definition of e^{A+B}, we have

$$e^{A+B} = \left(\sum_{j=0}^{\infty} \tfrac{1}{(2j)!} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(\sum_{j=0}^{\infty} \tfrac{1}{(2j+1)!} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} e + e^{-1} & e - e^{-1} \\ e - e^{-1} & e + e^{-1} \end{pmatrix}.$$

Thus, it is clear that

$$e^{A+B} \neq e^A e^B \neq e^B e^A.$$

5. (7 pts) a. With $A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ and $B = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$, it is easy to see that

$$AB = \begin{pmatrix} 0 & \alpha\beta \\ -\alpha\beta & 0 \end{pmatrix} = BA,$$

so $e^{A+B} = e^A e^B$.

b. From the definition of e^{Bt}, we have:

$$e^{Bt} = I + tB + \tfrac{t^2}{2!} B^2 + \cdots + \tfrac{t^k}{k!} B^k + \ldots$$

We have

$$B^2 = -\beta^2 I,$$

so it follows that

$$B^{2k} = (B^2)^k = (-1)^k \beta^{2k} I \qquad \text{and} \qquad B^{2k+1} = (B^2)^k \cdot B = (-1)^k \beta^{2k} B.$$

Thus, for e^{Bt}, the two diagonal elements are

$$1 - \tfrac{\beta^2 t^2}{2!} + \tfrac{\beta^4 t^4}{4!} - \cdots + \tfrac{(-1)^k \beta^{2k} t^{2k}}{(2k)!} + \cdots = \cos(\beta t),$$

and the upper diagonal element is

$$\beta t - \tfrac{\beta^3 t^3}{3!} + \tfrac{\beta^5 t^5}{5!} - \cdots + \tfrac{(-1)^k \beta^{2k+1} t^{2k+1}}{(2k+1)!} + \cdots = \sin(\beta t),$$

with the lower diagonal element being the negative of the upper diagonal element. This gives:

$$e^{Bt} = \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}.$$

c. From above, we have $e^{At} = e^{\alpha t} I$ and $e^{Bt} = \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}$ with

$$e^{(A+B)t} = e^{At}e^{Bt} = e^{\alpha t} \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix} = \begin{pmatrix} e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ -e^{\alpha t}\sin(\beta t) & e^{\alpha t}\cos(\beta t) \end{pmatrix},$$

which is the real Jordan canonical form e^{Jt}.

6. (6pts) a. (Proof of Abel's formula for 2×2 case) If $b_{ij}(t)$ is differentiable for

$$B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix} \qquad \text{and} \qquad \det(B(t)) = b_{11}(t)b_{22}(t) - b_{12}(t)b_{21}(t),$$

then

$$\begin{aligned} \frac{d}{dt}\left[\det(B(t))\right] &= \left[b_{11}'(t)b_{22}(t) + b_{11}(t)b_{22}'(t)\right] - \left[b_{12}'(t)b_{21}(t) + b_{12}(t)b_{21}'(t)\right], \\ &= \left[b_{11}'(t)b_{22}(t) - b_{12}'(t)b_{21}(t)\right] + \left[b_{11}(t)b_{22}'(t) - b_{12}(t)b_{21}'(t)\right], \\ &= \det\begin{vmatrix} b_{11}'(t) & b_{12}'(t) \\ b_{21}(t) & b_{22}(t) \end{vmatrix} + \det\begin{vmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}'(t) & b_{22}'(t) \end{vmatrix}. \end{aligned}$$

b. Define

$$\mathbf{\Phi}(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix} \qquad \text{and} \qquad A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix},$$

where $\mathbf{\Phi}(t)$ is a fundamental solution of $x' = A(t)x$. From Part a, we have:

$$\frac{d}{dt}\left(\det \mathbf{\Phi}(t)\right) = \det\begin{vmatrix} x_{11}'(t) & x_{12}'(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} + \det\begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}'(t) & x_{22}'(t) \end{vmatrix}.$$

Since $x' = A(t)x$, we have componentwise

$$x_{11}' = a_{11}x_{11} + a_{12}x_{21} = \sum_{k=1}^{2} a_{1k}x_{k1}, \quad \text{etc.}$$

so the previous expression satisfies:

$$
\begin{aligned}
\frac{d}{dt}\left(\det \boldsymbol{\Phi}\right) &= \det\begin{vmatrix} a_{11}x_{11}+a_{12}x_{21} & a_{11}x_{12}+a_{12}x_{22} \\ x_{21} & x_{22} \end{vmatrix} + \det\begin{vmatrix} x_{11} & x_{12} \\ a_{21}x_{11}+a_{22}x_{21} & a_{21}x_{12}+a_{22}x_{22} \end{vmatrix}, \\
&= x_{22}\Big[a_{11}x_{11}+a_{12}x_{21}\Big] - x_{21}\Big[a_{11}x_{12}+a_{12}x_{22}\Big] \\
&\quad + x_{11}\Big[a_{21}x_{12}+a_{22}x_{22}\Big] - x_{12}\Big[a_{21}x_{11}+a_{22}x_{21}\Big], \\
&= a_{11}\Big[x_{11}x_{22}-x_{12}x_{21}\Big] + a_{22}\Big[x_{11}x_{22}-x_{12}x_{21}\Big], \\
&= a_{11}\det(\boldsymbol{\Phi}) + a_{22}\det(\boldsymbol{\Phi}) \;=\; \sum_{i=1}^{2} a_{ii}\det(\boldsymbol{\Phi}) \;=\; \mathrm{tr}(A)\det(\boldsymbol{\Phi}).
\end{aligned}
$$

c. With $z(t) = \det \boldsymbol{\Phi}(t)$ where $\boldsymbol{\Phi}(t)$ is the fundamental solution to $x' = A(t)x$, Part b gave:

$$z' = \mathrm{tr}(A(t))z, \qquad \text{with} \quad z(0) = \det \boldsymbol{\Phi}(0),$$

which is a linear, homogeneous scalar equation. This equation is readily solved by integration to give:

$$z(t) = z(0)e^{\int_0^t (\mathrm{tr}A(s))ds},$$

which establishes this special case of Abel's formula.

7. (9pts) For the linear system of ODEs given by $\dot{\mathbf{x}} = A\mathbf{x}$, where

$$A = \begin{pmatrix} 4 & 6 & -15 \\ 1 & 3 & -5 \\ 1 & 2 & -4 \end{pmatrix}, \qquad \mathbf{x}_0 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

the characteristic equation satisfies:

$$\det|A-\lambda| = \begin{vmatrix} 4-\lambda & 6 & -15 \\ 1 & 3-\lambda & -5 \\ 1 & 2 & -4-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda-1)^3 = 0,$$

which gives $\lambda = 1$ an eigenvalue with algebraic multiplicity of 3. Computing $A - \lambda I$ gives

$$A - I = \begin{pmatrix} 3 & 6 & -15 \\ 1 & 2 & -5 \\ 1 & 2 & -5 \end{pmatrix} \qquad \text{and} \qquad (A-I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which implies the null space (kernel) or eigenspace has dimension 2. Since $\mathbf{v}_2 = [1,0,0]^T$ is not an element of the null space, we employ the Jordan chain method to find $\mathbf{v}_1$ by solving $(A-I)\mathbf{v}_2 = \mathbf{v}_1$. It is easy to see that

$$(A-I)\mathbf{v}_2 = \begin{pmatrix} 3 & 6 & -15 \\ 1 & 2 & -5 \\ 1 & 2 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \mathbf{v}_1.$$

The remaining eigenvector must solve $(A - I)\mathbf{v}_3 = \mathbf{0}$ and be linearly independent of $\mathbf{v}_1$. It is not hard to see that

$$(A - I)\mathbf{v}_3 = \begin{pmatrix} 3 & 6 & -15 \\ 1 & 2 & -5 \\ 1 & 2 & -5 \end{pmatrix} \mathbf{v}_3 = \mathbf{0} \quad \text{is satisfied by} \quad \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}.$$

From the eigenvectors we create the transition matrix (and inverse):

$$P = \begin{pmatrix} 3 & 1 & 5 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad P^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -5 \\ 0 & -1 & 1 \end{pmatrix}.$$

We obtain the Jordan canonical form, J, and solution, e^{Jt} (solving $\dot{\mathbf{y}} = J\mathbf{y}$):

$$J = P^{-1}AP = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{\Psi}(t) = e^{Jt} = \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix}.$$

The fundamental solution satisfies:

$$\mathbf{\Phi}(t) = e^{At} = Pe^{Jt}P^{-1} = e^t \begin{pmatrix} 3t+1 & 6t & -15t \\ t & 1+2t & -5t \\ t & 2t & -5t+1 \end{pmatrix}.$$

For the linear system of ODEs given by $\dot{\mathbf{x}} = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -6 & -7 & -4 \end{pmatrix},$$

A is a **companion matrix** (or the transpose depending on definition). The characteristic equation satisfies:

$$\det|A - \lambda| = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ -2 & -6 & -7 & -4-\lambda \end{vmatrix} = \lambda^4 + 4\lambda^3 + 7\lambda^2 + 6\lambda + 2 = (\lambda^2 + 2\lambda + 2)(\lambda + 1)^2 = 0,$$

which gives $\lambda_{1,2} = -1$ an eigenvalue with algebraic multiplicity of 2 and complex $\lambda_c = -1 \pm i$. Computing $A - \lambda_1 I$ gives

$$A + I = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & -6 & -7 & -3 \end{pmatrix},$$

which is a matrix with rank $= 3$, implies the null space (kernel) or eigenspace for $\lambda_1 = -1$ only has dimension 1. If the first entry is $\mathbf{v}_1 = 1$, then solving

$$(A + I)\mathbf{v}_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & -6 & -7 & -3 \end{pmatrix} \mathbf{v}_1 = \mathbf{0} \qquad \text{gives} \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

The **companion matrix** is easily seen to have eigenvectors of the form $\mathbf{v} = [1, \lambda, \lambda^2, ..., \lambda^{n-1}]^T$. The higher null space eigenvector, $\mathbf{v}_2$, solves $(A+I)\mathbf{v}_2 = \mathbf{v}_1$, so again if the first entry $\mathbf{v}_2 = 1$, then it is not hard to see that

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & -6 & -7 & -3 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad \text{so} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}.$$

We find an eigenvector associated with $\lambda_c = -1+i$, which from our statement above about companion matrices gives:

$$\begin{pmatrix} -\lambda_c & 1 & 0 & 0 \\ 0 & -\lambda_c & 1 & 0 \\ 0 & 0 & -\lambda_c & 1 \\ -2 & -6 & -7 & -3-\lambda_c \end{pmatrix} \mathbf{v}_3 = \mathbf{0}, \qquad \text{so} \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ \lambda_c \\ \lambda_c^2 \\ \lambda_c^3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1+i \\ -2i \\ 2+2i \end{pmatrix}$$

From the class notes, the real Jordan form matrix with block diagonal anti-symmetric matrices for complex eigenvalues are obtained by using the real and imaginary parts of any complex eigenvector. Thus, for $\mathbf{v}_3 = \mathbf{u}_3 + i\mathbf{w}_3$ with $\mathbf{u}_3$ and $\mathbf{w}_3$ real, we can now obtain our transition matrix $P = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}_3, \mathbf{w}_3]$ or

$$P = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & 1 \\ 1 & -1 & 0 & -2 \\ -1 & 2 & 2 & 2 \end{pmatrix} \qquad \text{and} \qquad P^{-1} = \begin{pmatrix} 0 & -2 & -2 & -1 \\ 2 & 4 & 3 & 1 \\ -1 & -2 & -1 & 0 \\ -1 & -3 & -3 & -1 \end{pmatrix}.$$

We obtain the Jordan canonical form, J, and solution, e^{Jt} (solving $\dot{\mathbf{y}} = J\mathbf{y}$):

$$J = P^{-1}AP = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{\Psi}(t) = e^{Jt} = \begin{pmatrix} e^{-t} & te^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{-t}\cos(t) & e^{-t}\sin(t) \\ 0 & 0 & -e^{-t}\sin(t) & e^{-t}\cos(t) \end{pmatrix}.$$

8. (7pts) a. For real parameters a, b, and c, we consider the matrix

$$A = \begin{pmatrix} a & 0 & 0 & a \\ 0 & a & b & 0 \\ 0 & c & a & 0 \\ a & 0 & 0 & a \end{pmatrix} = \begin{pmatrix} aI_2 & A_{12} \\ A_{21} & aI_2 \end{pmatrix},$$

where

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_{12} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \qquad A_{21} = \begin{pmatrix} 0 & c \\ a & 0 \end{pmatrix}.$$

The eigenvalues are found by solving:

$$\begin{aligned} \det \begin{vmatrix} (a-\lambda)I_2 & A_{12} \\ A_{21} & (a-\lambda)I_2 \end{vmatrix} &= \det \left|(a-\lambda)^2 I_2 - A_{12}A_{21}\right| \\ &= \det \begin{vmatrix} (a-\lambda)^2 - a^2 & 0 \\ 0 & (a-\lambda)^2 - bc \end{vmatrix} \\ &= \left((a-\lambda)^2 - a^2\right)\left((a-\lambda)^2 - bc\right) \;=\; 0. \end{aligned}$$

It follows that the eigenvalues are

$$\lambda = a \pm a = 0, 2a, \qquad \text{and} \qquad \lambda = a \pm \sqrt{bc}.$$

1. If $a \neq 0$, $bc > 0$, and $bc \neq a^2$, then there are 4 distinct real eigenvalues.
2. If $a \neq 0$, $bc > 0$, and $bc = a^2$, then the eigenvalues are $\lambda = 0, 0, 2a, 2a$, a pair of repeated real eigenvalues.
3. If $a \neq 0$ and $bc < 0$, then the eigenvalues are $\lambda = 0, 2a$ (real) and two complex $\lambda = a \pm i\sqrt{|bc|}$.
4. If $a \neq 0$ and $bc = 0$, then the eigenvalues are $\lambda = 0, a, a, 2a$ (real) with $\lambda = a$ repeated.
5. If $a = 0$, $bc > 0$, and $bc \neq a^2$, then $\lambda = 0, 0, \pm\sqrt{bc}$, which are 3 distinct eigenvalues with $\lambda = 0$ repeated.
6. If $a = 0$ and $bc < 0$, then there are the repeated real eigenvalues $\lambda = 0, 0$, and a pair of imaginary $\lambda = \pm i\sqrt{|bc|}$.
7. If $a = 0$ and $bc = 0$, then $\lambda = 0$ is the only eigenvalue with algebraic multiplicity of 4.

b. If $a = b = c = 2$, then we are in Case 2 with two pairs of repeated eigenvalues, $\lambda = 0, 0, 4, 4$. For $\lambda = 0$, we solve

$$(A - 0I)\mathbf{v} = A\mathbf{v} = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} \mathbf{v} = \mathbf{0},$$

which has geometric multiplicity **2** and gives

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

For $\lambda = 4$, we solve

$$(A - 4I)\mathbf{v} = \begin{pmatrix} -2 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0},$$

which has geometric multiplicity **2** and gives

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

The transformation matrix and its inverse are

$$P = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad P^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

so the Jordan canonical form becomes:

$$P^{-1}AP = J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

It readily follows that for $\dot{\mathbf{\Psi}} = J\mathbf{\Psi}$, the **fundamental solution** satisfies:

$$\mathbf{\Psi}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{4t} & 0 \\ 0 & 0 & 0 & e^{4t} \end{pmatrix}.$$

9. (2pts) If A is an invertible matrix, then we know $AA^{-1} = I$ and $\|I\| = 1$. By the sub-multiplicative property of norms:

$$\|A\|\|A^{-1}\| \geq \|AA^{-1}\| = \|I\| = 1.$$