Fall 2021 **Homework 3 – Fundamental Solutions** Math 537

Due Fri. 10/1 in Gradescope

Work the **4** problems in WeBWorK and entering written parts in Gradescope.

WW1 (5pts) Transformation and phase portraits (2).

WW2 (5pts) Transformation and phase portraits (2).

WW3 (5pts) Transformation and phase portraits (2).

WW4 (5pts) Transformation and phase portraits (2).

1. (6pts) a. Use the matrix definition of e^{At} to find a fundamental matrix solution of $\dot{\mathbf{y}} = A\mathbf{y}$, where

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}.$$

b. Use the matrix definition of e^{At} to find a fundamental matrix solution of $\dot{\mathbf{y}} = A\mathbf{y}$, where A is the $n \times n$ matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}.$$

2. (3pts) Use the matrix definition of e^A to show that if P is a nonsingular $n \times n$ matrix,

$$P^{-1}e^AP = e^{P^{-1}AP}.$$

3. (4pts) Show that if $AB = BA$ (commute), then

$$e^{A+B} = e^Ae^B = e^Be^A.$$

Since A and B commute, you can make use of the binomial theorem which says

$$(A+B)^m = \sum_{l=0}^{m} \binom{m}{l} A^lB^{m-l}, \quad \binom{m}{l} = \frac{m!}{l!(m-l)!}$$

You will also need to use a change of order for the double sum that will come up. This means using the identity

$$\sum_{j=0}^{\infty}\sum_{l=0}^{j} a_{jl} = \sum_{l=0}^{\infty}\sum_{j=l}^{\infty} a_{jl}.$$

4. (4pts) Find an example of two matrices such that

$$e^{A+B} \neq e^A e^B.$$

Hint: You can do this with 2×2 matrices.

5. (7pts) Consider

$$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = A + B, \quad \text{and} \quad A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

a. Show that A and B commute, so $e^{A+B} = e^A e^B$.

b. Use the definition of e^{Bt} to prove that

$$e^{Bt} = \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}.$$

c. From these results, give the real Jordan canonical form, e^{Jt}.

6. (6pts) (Proof of Abel's formula for 2×2 case)
a. Let

$$B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix}$$

with $b_{ij}(t)$ differentiable. Compute $\frac{d}{dt}(\det B(t))$ by first expanding $\det B(t)$ and then differentiating. Next show that

$$\frac{d}{dt}(\det B(t)) = \det \begin{pmatrix} b'_{11}(t) & b'_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix} + \det \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b'_{21}(t) & b'_{22}(t) \end{pmatrix}.$$

b. Let

$$\mathbf{\Phi}(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix}$$

be a fundamental solution for $x' = A(t)x$, where

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}.$$

Show that

$$\begin{aligned} \frac{d}{dt}(\det \mathbf{\Phi}(t)) &= \det \begin{pmatrix} \sum_{k=1}^2 a_{1k} x_{k1}(t) & \sum_{k=1}^2 a_{1k} x_{k2}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix} + \det \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ \sum_{k=1}^2 a_{2k} x_{k1}(t) & \sum_{k=1}^2 a_{2k} x_{k2}(t) \end{pmatrix}. \\ &= \sum_{i=1}^{2} a_{ii} \det \mathbf{\Phi}(t). \end{aligned}$$

c. Let $z(t) = \det \mathbf{\Phi}(t)$ where $\mathbf{\Phi}(t)$ is the fundamental solution to $x' = A(t)x$ as above. Use Part b to conclude that

$$z(t) = z(0)e^{\int_0^t (\text{tr}A(s))ds},$$

thus establishing this special case of Abel's formula.

7. (9pts) Consider the linear system of ODEs given by

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0.$$

For each of the following matrices A, find a transition matrix P that transforms A into the **real Jordan canonical form**, J. Write both P and J. Furthermore, give a **fundamental solution**, $\mathbf{\Psi}(t) = e^{Jt}$.

$$A = \begin{pmatrix} 4 & 6 & -15 \\ 1 & 3 & -5 \\ 1 & 2 & -4 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -6 & -7 & -4 \end{pmatrix}.$$

Write the fundamental solution $\mathbf{\Phi}(t) = e^{At}$ for the 3×3 matrix A on the left.

8. (7pts) a. For real paramaters a, b, and c, describe the regions in (a, b, c) space where the matrix

$$A = \begin{pmatrix} a & 0 & 0 & a \\ 0 & a & b & 0 \\ 0 & c & a & 0 \\ a & 0 & 0 & a \end{pmatrix}$$

has real, complex, and repeated eigenvalues. **Hint:** Breaking A up into four 2×2 sub-matrices, or blocks, makes your life significantly easier. So use

$$A = \begin{pmatrix} aI_2 & A_{12} \\ A_{21} & aI_2 \end{pmatrix}, \; I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; A_{12} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \; A_{21} = \begin{pmatrix} 0 & c \\ a & 0 \end{pmatrix}.$$

b. Let $a = b = c = 2$. Transform A into the **real Jordan canonical form**, J, writing both P and J. Then give a **fundamental solution**, $\mathbf{\Psi}(t) = e^{Jt}$, solving the system of ODEs

$$\dot{\mathbf{\Psi}} = J\mathbf{\Psi}.$$

9. (2pts) If A is an invertible matrix, show that

$$\|A\|\|A^{-1}\| \geq 1.$$