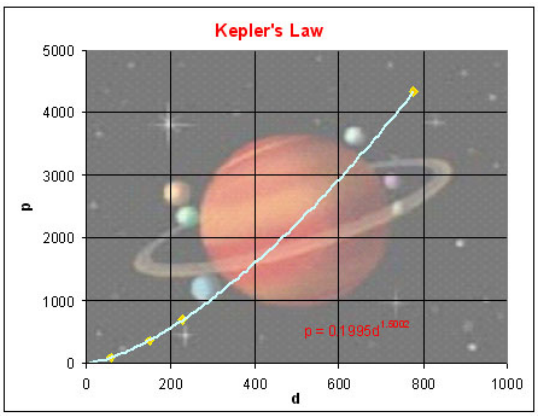
Homework 1 – Linear ODEs and Allometric Solutions

Written problem in WeBWorK.

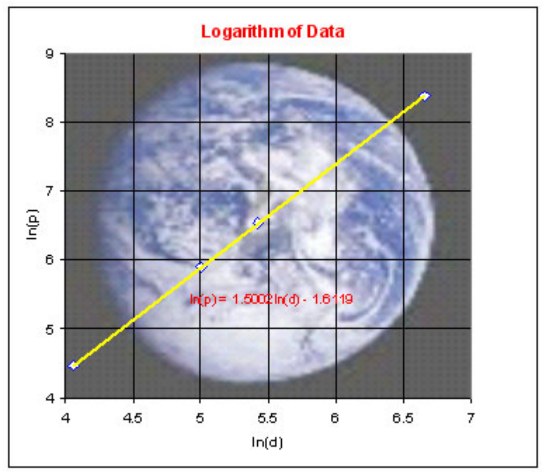
4. b. (4pts) The graph below gives the best power law fit, showing $k = 0.1995$ and $a = 1.5002$, so

$$p = 0.1995\, d^{1.5002}.$$

The power law clearly fits the data very well.



Kepler model



Linear model of ln of data

The power in the allometric model matches the slope of the logarithmic model, while the coefficient of the allometric model is the exponential of the intercept of the linear model of the logarithms of the data.

d. (1pts) When checking the model against the Jet Propulsion Laboratory data, one finds that the errors are extremely small. This is not a surprise as Kepler's law is used to help compute the distance and period.

e. (5pts) We set the gravitational force, F_g, equal to the centripetal force, F_c,

$$\begin{aligned} \frac{GMm}{d^2} &= \frac{mv^2}{d} \\ v^2 &= \frac{GM}{d}. \end{aligned}$$

However, $v = 2\pi d/p$, where p is the period of the planet around the Sun. It follows that

$$\begin{aligned} \left(\frac{2\pi d}{p}\right)^2 &= \frac{GM}{d} \\ p &= \frac{2\pi d^{3/2}}{\sqrt{GM}}. \end{aligned}$$

Thus, the power of d fits the power law obtained by in Part a. If one uses the universal gravitational constant $G = 6.673 \times 10^{-11}$ m^{3}kg^{-1}s^{-2} and the mass of the Sun, $M = 1.98892 \times 10^{30}$ and converts to correct units, one obtains the constant k found by fitting the data.

1. (5pts) For the linear ODE

$$\frac{dy}{dx} + 2xy = f(x), \quad y(0) = 2, \qquad \text{where} \quad f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}$$

we find the integrating factor

$$\mu(x) = e^{\int_0^x 2x\,dx} = e^{x^2}.$$

Thus,

$$\frac{d}{dx}\left(y(x)e^{x^2}\right) = f(x)e^{x^2},$$

so

$$y(x)e^{x^2} - 2 = \int_0^x f(s)e^{s^2}ds \quad \text{or} \quad y(x) = e^{-x^2}\left(2 + \int_0^x f(s)e^{s^2}ds\right).$$

If $x \in [0,1]$, then

$$\int_0^x f(s)e^{s^2}ds = \int_0^x s\,e^{s^2}ds = \tfrac{1}{2}\left(e^{x^2} - 1\right).$$

For $x > 1$,

$$\int_0^x f(s)e^{s^2}ds = \int_0^1 s\,e^{s^2}ds + \int_1^x 0 \cdot e^{s^2}ds = \tfrac{1}{2}(e - 1).$$

It follows that

$$\begin{aligned} y(x) &= e^{-x^2}\left(2 + \tfrac{1}{2}H(x)\right) \quad \text{where} \quad H(x) = \begin{cases} e^{x^2} - 1, & 0 \le x < 1 \\ e - 1, & x > 1 \end{cases}, \\ y(x) &= \begin{cases} \frac{3}{2}e^{-x^2} + \frac{1}{2}, & 0 \le x < 1 \\ \left(\frac{3}{2} + \frac{e}{2}\right)e^{-x^2}, & x > 1 \end{cases} \end{aligned}$$

2. (5pts) The ODE,

$$\dot{x} + p(t)x = 0 \qquad \text{with} \quad x(0) = x_0,$$

has the solution:

$$x(t) = x_0 e^{-\int_0^t p(s)ds},$$

so

$$\begin{aligned} x(t+T) &= x_0 e^{-\int_0^{t+T} p(s)ds} = x_0 e^{-\int_0^T p(s)ds - \int_T^{T+t} p(s)ds}, \\ &= \left(e^{-\int_0^T p(s)ds}\right) x_0 e^{-\int_0^t p(z+T)dz}, \\ &= \left(e^{-\int_0^T p(s)ds}\right) x_0 e^{-\int_0^t p(z)dz}, \\ &= \left(e^{-\int_0^T p(s)ds}\right) x(t). \end{aligned}$$

It follows that $x(t+T) = x(t)$ if and only if

$$e^{-\int_0^T p(s)ds} = 1, \quad \text{which is equivalent to} \quad \int_0^T p(s)ds = 0.$$

Thus, periodic $p(t)$ has zero average in time is equivalent to the solution being periodic.