Take-Home Final Due Thur. 12/16, Noon

For full credit you must show details of the steps you take. Computer solutions are allowed for basic operations like finding a matrix inverse, multiplying matrices, or integrations, but write explicitly what you did.

1. (20pts) Consider the following ODE:

$$xy'' - y' + 4x^3y = 0. \tag{1}$$

a. Show that $x = 0$ is a regular singular point, find the indicial equation and recurrence relations, and determine the two linearly independent solutions. Briefly explain whether or not this example requires a logarithmic element for the second solution, *i.e.*, provide some justification for which case in the method of Froebenius you are using and some specifics about the case for this example. Determine expressions for the coefficients for these two solutions.

b. In Part a, $y_1(x)$ should be easily recognizable as a basic function. One of the earlier homework problems introduced the **reduction of order** method. Use this technique to find $y_2(x)$ for (1) and compare this solution to your series solution in Part a.

2. (20pts) a. Use singular perturbation methods to obtain a uniform approximation to the solution of the BVP:

$$\varepsilon y'' + 2y' + e^y = 0, \qquad y(0) = 0, \quad y(1) = 0, \quad 0 < \varepsilon \ll 1.$$

State clearly both the inner and outer solutions that you derived.

b. Provide computer simulations of these solutions for $\varepsilon = 0.1$, 0.05, and 0.01. Briefly discuss the observed behavior and explain what happens to the inner, outer, and uniform solutions as $\varepsilon \to 0$.

3. (20pts) a. Use singular perturbation methods to obtain a uniform approximation to the solution of the IVP:

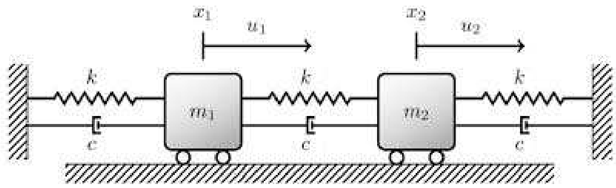
$$\frac{du}{dt} = v, \qquad \varepsilon\frac{dv}{dt} = -u^2 - v, \qquad u(0) = 1, \quad v(0) = 0, \quad 0 < \varepsilon \ll 1.$$

State clearly both the inner and outer solutions that you derived for both $u(t)$ and $v(t)$.

b. Provide computer simulations of these solutions for $\varepsilon = 0.1$, 0.05, and 0.01 with $t \in [0, 5]$. Briefly discuss the observed behavior and explain what happens to the inner, outer, and uniform solutions as $\varepsilon \to 0$.

4. (30pts) Consider the mass-spring problem, where we consider two identical masses connected by three identical springs. Assuming a general Hooke's law spring, using Newton's law of forces, and

ignoring the viscous damping between the two springs, the following system of second order linear ODEs can be written:



$$\begin{aligned} m\ddot{x}_1 &= -c\dot{x}_1 - kx_1 + k(x_2 - x_1), \\ m\ddot{x}_2 &= -c\dot{x}_2 - kx_2 + k(x_1 - x_2), \end{aligned}$$

where $u_1 = \dot{x}_1$ and $u_2 = \dot{x}_2$.

a. For this part of the problem, we assume no damping, so $c = 0$. Define the **natural frequency** by the constant $\omega^2 = \frac{k}{m}$. Define new state variables $\mathbf{y} = [y_1, y_2, y_3, y_4]^T$, where $y_1 = x_1$, $y_2 = u_1$, $y_3 = x_2$, and $y_4 = u_2$. Rewrite the above system of second order linear ODEs into a system of first order linear ODEs:

$$\dot{\mathbf{y}} = A\mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{y}_0. \tag{2}$$

Transform A into J_a, where J_a is a matrix in **real Jordan canonical form**. Show how you obtain your nonsingular transforming matrix, P, and give its inverse, P^{-1}. (Hint: You may want to use the theorem described on Slide 36 of the Fundamental Solutions lecture notes, which was added after the lecture.) Write the fundamental solution, $\mathbf{\Psi}(t) = e^{J_a t}$.

b. In lecture we noted that the eigenspaces of A are invariant subspaces for the flow, $\mathbf{\Phi}(t) = e^{At}$, dividing $\mathbb{R}^4$ into E^s (stable), E^u (unstable), and E^c (center) subspaces. With $c = 0$, give the dimension of each of these subspaces for this example. What is the equilibrium point and is it hyperbolic? Briefly explain these implications for the qualitative behavior of this system. Are there limitations to this theory for this particular model?

c. Again with $c = 0$, assume this system is initially at rest, and the two masses are displaced by x_{10} and x_{20}, respectively, *i.e.*, we have:

$$\mathbf{y}(0) = \left(x_{10}, 0, x_{20}, 0\right)^T.$$

Write the unique solution to this initial value problem, $\mathbf{y}(t)$. Briefly discuss the solution describing the motion when the two masses are equally displaced to the right, $x_{10} = x_{20} > 0$ (symmetric motion). Also, briefly discuss the solution describing the motion when the two masses are equally displaced in opposite directions, $x_{10} = -x_{20} > 0$ (antisymmetric motion).

d. For this part of the problem, we assume damping, $c > 0$. Define $\omega^2 = \frac{k}{m}$, $2\gamma = \frac{c}{m} \ll \omega$, and new state variables $\mathbf{y} = [y_1, y_2, y_3, y_4]^T$, where $y_1 = x_1$, $y_2 = u_1$, $y_3 = x_2$, and $y_4 = u_2$. Rewrite the above system of second order linear ODEs into a system of first order linear ODEs:

$$\dot{\mathbf{y}} = B\mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{y}_0. \tag{3}$$

Transform B into J_b, where J_b is a matrix in **real Jordan canonical form**. Show how you obtain your nonsingular transforming matrix, P. Write the fundamental solution, $\mathbf{\Psi}(t) = e^{J_b t}$.

e. The eigenspaces of B are invariant subspaces for the flow, $\mathbf{\Phi}(t) = e^{Bt}$. With $c > 0$ and $2\gamma = \frac{c}{m} \ll \omega$, give the dimension of each of these eigenspaces for this example. What is the equilibrium point and is it hyperbolic? Briefly explain these implications for the qualitative behavior of this system. Are there limitations to this theory for this particular model?

5. (25pts) This problem examines periodic forcing of the basic harmonic oscillator, using both the **variation of parameters method** and two forms of **regular perturbation methods**.

a. Consider the second order linear ODE given by:

$$\ddot{y} + y = \varepsilon \sin(\omega t), \qquad y(0) = 1, \quad \dot{y}(0) = 0,$$

where $0 < \varepsilon \ll 1$ and ω are two positive parameters. Let $y(t) = x_1(t)$ and $\dot{y} = x_2(t)$. Transform this second order nonhomogeneous linear ODE into a system of first order linear ODEs:

$$\dot{\mathbf{x}} = A\mathbf{x} + \varepsilon f(t), \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where A is in **real Jordan canonical form** and $f(t)$ is the appropriate forcing function. Write the fundamental solution, $\mathbf{\Phi}(t) = e^{At}$. Use $\mathbf{\Phi}(t)$ and the **variation of parameters method** to find your unique solution $\mathbf{x}(t)$ to the IVP above. Be sure to note any special cases.

b. Now consider the second order nonlinear ODE given by:

$$\ddot{y} + y = \varepsilon y\left(1 - \dot{y}^2\right), \qquad y(0) = 1, \quad \dot{y}(0) = 0, \tag{4}$$

where $0 < \varepsilon \ll 1$. Once again the right hand side is a small periodic forcing term. Use a **regular perturbation method** to find an approximate solution to this nonlinear problem. Determine an expansion to $\mathcal{O}\left(\varepsilon^2\right)$, *i.e.*, find a two term ε expansion of $y(t)$. Is this approximate solution bounded? Explain.

c. In this part we again consider the second order nonlinear IVP given in Part b, (4). However, this time we find an approximation using the **Poincaré-Lindstedt perturbation method**. You rescale the time, t, in an ε expansion ($\tau = \omega t$ with $\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots$) along with ε expansion of $y(\tau)$. Apply the Poincaré-Lindstedt perturbation method to (4) and remove any **secular terms** in your order ε term for the $y(\tau)$ approximation. Give your two term approximate solution $y(\tau)$ along with your two term time scaling τ. Is this approximate solution necessarily bounded? Explain.

d. Let $\varepsilon = 0.1$ and 0.02 and use a numerical differential equation solver (like MatLab's ODE45) to find an accurate numerical solution of (4) for $t \in [0, 50]$. Create a graph comparing the **regular perturbation approximation** of Part b, the **Poincaré-Lindstedt perturbation approximation** of Part c, and the numerical solution. Briefly describe what you observe and how these various methods compare.