1. (10pts) Consider the differential equation

$$\varepsilon \frac{dy}{dt} + y = f(t/\varepsilon), \quad y(0) = y_0, \qquad t \geq 0,$$

where $f(t)$ satisfies $f(t+T) = f(t)$, and on $[0, T]$, $f(t)$ is defined to be

$$f(t) = \begin{cases} 1 & 0 \leq t < T/2, \\ 0 & T/2 \leq t < T. \end{cases}$$

We rescale this problem by letting $\xi = \frac{t}{\varepsilon}$. It follows that $\varepsilon \frac{dy}{dt} = \varepsilon \frac{dy}{d\xi}\frac{d\xi}{dt} = \frac{dy}{d\xi}$, so the scaled ODE becomes:

$$\frac{dy}{d\xi} + y = f(\xi), \qquad y(0) = y_0. \tag{1}$$

This has the integrating factor $\mu(\xi) = e^{\xi}$, so

$$\frac{d}{d\xi}\left[e^{\xi} y(\xi)\right] = e^{\xi} f(\xi), \quad \text{or} \quad e^{\xi} y(\xi) - y_0 = \int_0^{\xi} e^s f(s) ds.$$

Thus, the solution is given by:

$$y(\xi) = e^{-\xi}\left(y_0 + \int_0^{\xi} e^s f(s) ds\right),$$

which is readily integrated. However, since $f(\xi)$ is discontinuous (series of step functions), this problem is most amenable to solution by Laplace transforms. Let $Y(s) = \mathcal{L}[y(\xi)]$, then transforming (1) gives:

$$sY(s) - y_0 + Y(s) = \mathcal{L}[f(\xi)] \qquad \text{or} \qquad (s+1)Y(s) = y_0 + \mathcal{L}[f(\xi)].$$

A theorem for periodic functions with period T states that

$$\begin{aligned} \mathcal{L}[f(\xi)] &= \frac{1}{1 - e^{-sT}} \int_0^T e^{-s\xi} f(\xi) d\xi = \frac{1}{1 - e^{-sT}} \int_0^{T/2} e^{-s\xi} d\xi, \\ &= \frac{1}{1 - e^{-sT}} \cdot \frac{1 - e^{-sT/2}}{s} = \frac{1}{s(1 + e^{-sT/2})}. \end{aligned}$$

It follows that

$$\begin{aligned} Y(s) &= \frac{y_0}{s+1} + \frac{1}{s(s+1)} \cdot \frac{1}{1 + e^{-sT/2}} = \frac{y_0}{s+1} + \frac{1}{1 + e^{-sT/2}}\left(\frac{1}{s} - \frac{1}{s+1}\right), \\ &= \frac{y_0}{s+1} + \sum_{n=0}^{\infty} (-1)^n e^{-snT/2}\left(\frac{1}{s} - \frac{1}{s+1}\right). \end{aligned}$$

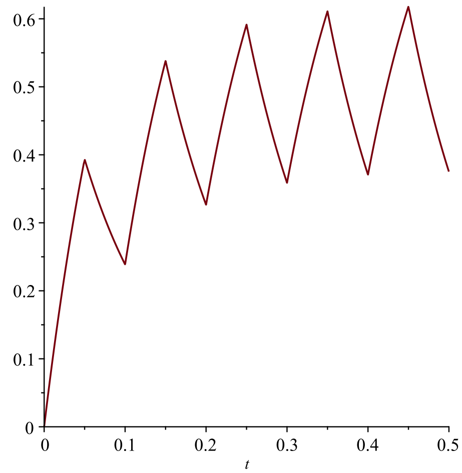
One takes the inverse Laplace transform and obtains the solution:

$$y(\xi) = y_0 e^{-\xi} + \sum_{n=0}^{\infty} (-1)^n u_{nT/2}(\xi)\left(1 - e^{-(\xi - nT/2)}\right).$$

Thus, the original ODE has the solution:

$$y(t) = y_0 e^{-t/\varepsilon} + \sum_{n=0}^{\infty} (-1)^n u_{nT/2}(t/\varepsilon) \left(1 - e^{-(t/\varepsilon - nT/2)}\right).$$

This previous expression readily shows that the solution $y(t)$ is not T-periodic, as $y(t) \neq y(t+T)$ for all y_0. However there exists a specific $y_0 \approx 0.377541$ that gives a periodic solution. This is shown with the Method of Averaging and a fixed point theorem.



2. (10pts) Show that if $||A|| < 1$, then one has that $(I - A)^{-1}$ exists and

$$\left|\left|(I - A)^{-1}\right|\right| \leq \frac{1}{1 - ||A||}.$$

Proof: We prove by contradiction that $(I - A)^{-1}$ exists. If $(I - A)$ is singular, then there exists a vector x such that $(I - A)x = 0$ or $x = Ax$. Taking norms we have

$$||x|| = ||Ax|| \leq ||A|| \, ||x|| \qquad \text{or} \qquad ||A|| \geq 1, \quad \text{as} \quad ||x|| \neq 0,$$

which is a contradiction and $(I - A)^{-1}$ exists.

Define

$$\begin{aligned} \tilde{A}_N &= (I - A) \sum_{j=0}^{N} A^j - I = \sum_{j=0}^{N} A^j - \sum_{j=0}^{N} A^{j+1} - I, \\ &= \sum_{j=0}^{N} A^j - \sum_{j=0}^{N+1} A^j = -A^{N+1} \end{aligned}$$

It follows that

$$\left|\left|\tilde{A}_N\right|\right| = \left|\left|-A^{N+1}\right|\right| = \left|\left|A^{N+1}\right|\right| \leq ||A||^{N+1}.$$

Since $||A|| < 1$, we have $||A||^{N+1} \to 0$ as $N \to \infty$, which implies that $\left|\left|\tilde{A}_N\right|\right| = 0$ as $N \to \infty$. It follows that

$$\lim_{N\to\infty} \tilde{A}_N = 0, \qquad \text{so} \qquad I = (I - A) \sum_{j=0}^{\infty} A^j.$$

This is equivalent to

$$(I - A)^{-1} = \sum_{j=0}^{\infty} A^j.$$

Taking norms and using the triangle inequality, we have

$$\left|\left|(I - A)^{-1}\right|\right| = \left|\left|\sum_{j=0}^{\infty} A^j\right|\right| \le \sum_{j=0}^{\infty} \left|\left|A^j\right|\right| \le \sum_{j=0}^{\infty} \left|\left|A\right|\right|^j.$$

With $||A|| < 1$, we have a geometric series, so

$$\left|\left|(I - A)^{-1}\right|\right| \le \sum_{j=0}^{\infty} \left|\left|A\right|\right|^j = \frac{1}{1 - ||A||}. \qquad q.e.d.$$

3. (15pts) This problem examines the system of differential equations given by:

$$\dot{\mathbf{x}} = \begin{pmatrix} -3 & -1 \\ 3 - \alpha & -3 \end{pmatrix} \mathbf{x},$$

where α is a real parameter.

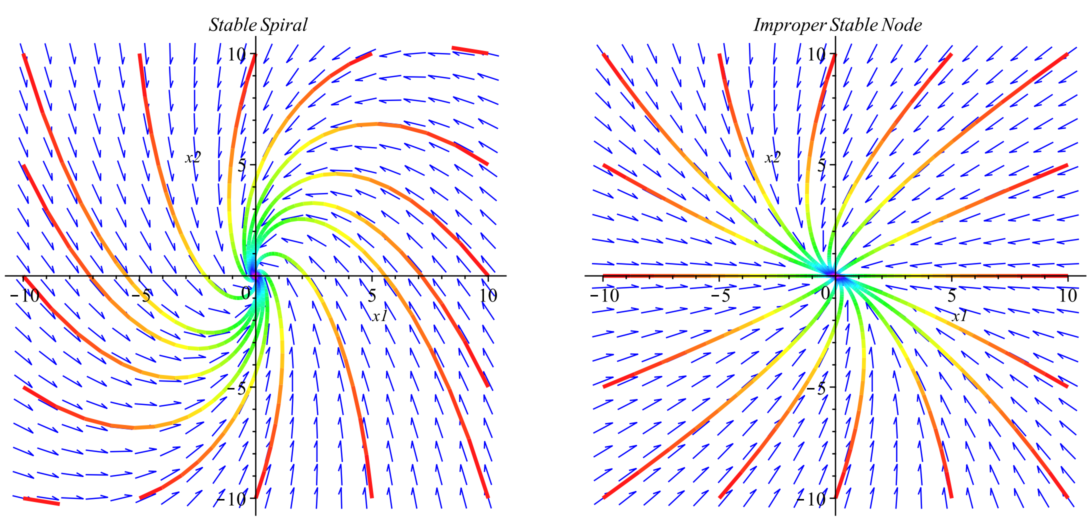
a. The characteristic equation satisfies:

$$\begin{vmatrix} -3 - \lambda & -1 \\ 3 - \alpha & -3 - \lambda \end{vmatrix} = (\lambda + 3)^2 + 3 - \alpha = 0.$$

It follows that $\lambda = -3 \pm \sqrt{\alpha - 3}$.

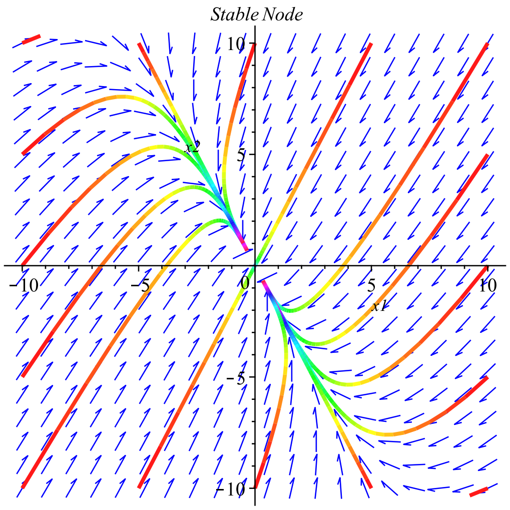
b and c. There are **two** critical values of α, where the qualitative nature of the phase portrait changes. When $\alpha = 3$, there is a change in the eigenvalues between being real and complex. The other critical value is when $\alpha = 12$, where one of the eigenvalues becomes positive.

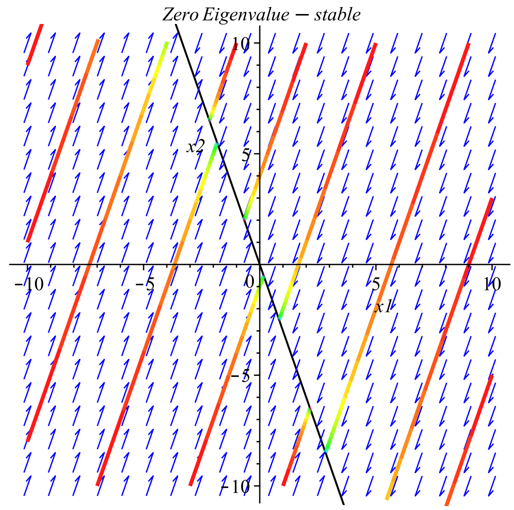
For $\alpha < 3$, the eigenvalues are complex with the real part less than zero, which results in a stable spiral. Below left shows the phase portrait for this region.



For $\alpha = 3$, the eigenvalues, $\lambda = -3$, are repeated roots, which results in an improper stable node. Above right shows the phase portrait for this value of α.

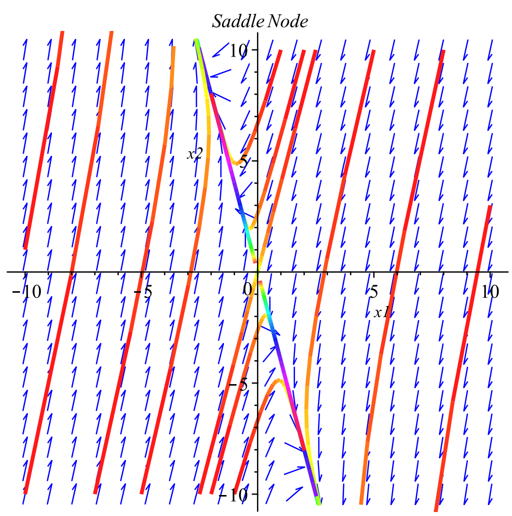
For $3 < \alpha < 12$, the eigenvalues are distinct real and less than zero, $\lambda_1 < \lambda_2 < 0$, which results in a stable node. Below left shows the phase portrait for this region.





For $\alpha = 12$, the eigenvalues, $\lambda = -6, 0$, are distinct roots. The eigenvalue, $\lambda_1 = 0$, results in a line of equilibria, $x_2 = -3x_1$, while $\lambda_2 = -6$ results in all solutions approaching (stable) these equilibria. Above right shows the phase portrait for this value of α.

For $\alpha > 12$, the eigenvalues are distinct real and opposite signs, $\lambda_1 < 0 < \lambda_2$, which results in a saddle node. Below shows the phase portrait for this region.



4. (10pts) a. Consider the linear system of ODEs given by

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -3 & 0 \\ -9 & 0 & -6 \end{pmatrix} \mathbf{x} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0.$$

The characteristic equation satisfies $\det|A - \lambda I| = 0$:

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -3-\lambda & 0 \\ -9 & 0 & -6-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -3-\lambda & 0 \\ 0 & -6-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & -3-\lambda \\ -9 & 0 \end{vmatrix}$$
$$= -\lambda(\lambda+3)(\lambda+6) - 9(\lambda+3) = -(\lambda+3)^3 = 0.$$

It follows that $\lambda = -3$ is an eigenvalue with algebraic multiplicity **3**. We examine $A + 3I$ and can see that this is a rank 1 matrix, so the $\ker(A + 3I)$ is two-dimensional, which implies the geometric multiplicity of $\lambda = -3$ is **2**. (Also, note that $(A+3I)^2 = 0$, giving the same geometric multiplicity.) It is easy to see that

$$(A+3I)\mathbf{v} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 0 & 0 \\ -9 & 0 & -3 \end{pmatrix} \mathbf{v} = 0 \quad \text{has eigenvectors} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

If we take $\mathbf{v}_2 = [1, 0, 0]^T$ to be in the generalized eigenspace of A, then the Jordan chain process gives:

$$(A+3I)\mathbf{v}_2 = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 0 & 0 \\ -9 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{v}_1, \quad \text{which gives} \quad \mathbf{v}_1 = \begin{pmatrix} 3 \\ 0 \\ -9 \end{pmatrix}.$$

With these eigenvectors, we obtain a transition matrix and with the help of Maple find its inverse:

$$P = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad P^{-1} = \begin{pmatrix} 0 & 0 & -\frac{1}{9} \\ 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 \end{pmatrix}$$

Subsequently, we obtain the Jordan canonical form:

$$J = P^{-1}AP = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

A fundamental solution satisfies:

$$\mathbf{\Psi}(t) = e^{Jt} = \begin{pmatrix} e^{-3t} & te^{-3t} & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix} = e^{-3t} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

b. Now consider the linear system of ODEs given by

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & 16 & -8 & 4 \end{pmatrix} \mathbf{x} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0.$$

This is a Vandermonde matrix, and with $y = x_1$, $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, and $\dot{x}_3 = x_4$, we obtain the 4^{th} order scalar ODE:

$$y'''' - 4y''' + 8y'' - 16y' + 16 = 0,$$

which has the characteristic equation:

$$\lambda^4 - 4\lambda^3 + 8\lambda^2 - 16\lambda + 16 = (\lambda - 2)^2(\lambda^2 + 4) = 0.$$

It follows that the eigenvalues are $\lambda = \pm 2i$,2,2. For the Vandermonde matrix, the eigenvalue $\lambda_1 = 2$ has algebraic multiplicity of **2**, but geometric multiplicity of **1** with $\mathbf{v}_1 = [1, 2, 4, 8]^T$. To find an eigenvector in the higher dimensional null space, we solve $(A - 2I)\mathbf{v}_2 = \mathbf{v}_1$ or:

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ -16 & 16 & -8 & 2 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix}, \qquad \text{so} \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 12 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix}.$$

Again because this is a Vandermonde matrix, the eigenvalue $\lambda_3 = 2i$ has the associated eigenvector $\mathbf{v}_3 = [1, 2i, -4, -8i]^T$ (with $\lambda_4 = -2i$ having associated eigenvector $\mathbf{v}_4 = \overline{\mathbf{v}_3}$). To obtain the real Jordan canonical form, we write $\mathbf{v}_3 = \mathbf{u} + i\mathbf{w}$, so

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ -4 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{w} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ -8 \end{pmatrix}.$$

It follows that an appropriate transition matrix and its inverse (from Maple) are:

$$P = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 4 & 4 & -4 & 0 \\ 8 & 12 & 0 & -8 \end{pmatrix} \qquad \text{and} \qquad P^{-1} = \begin{pmatrix} 1 & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{16} \\ -1 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{16} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{8} & 0 \end{pmatrix}.$$

It follows that the real Jordan canonical form is given by:

$$J = P^{-1}AP = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}.$$

A fundamental solution satisfies:

$$\mathbf{\Psi}(t) = e^{Jt} = \begin{pmatrix} e^{2t} & te^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & \cos(2t) & \sin(2t) \\ 0 & 0 & -\sin(2t) & \cos(2t) \end{pmatrix}.$$

5. a. For the nonhomogeneous system of linear ODEs:

$$\dot{\mathbf{x}} = A\mathbf{x} + g(t) = \begin{pmatrix} -\alpha & 1 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{-\gamma t} \\ 0 \\ 0 \\ \sin(\omega t) \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 4 \\ -2 \end{pmatrix},$$

with $\alpha, \beta, \gamma, \omega > 0$, we see that the matrix, A, given below is in real Jordan canonical form:

$$A = \begin{pmatrix} -\alpha & 1 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}.$$

Thus, the fundamental matrix solution of the homogeneous part of the ODE is given by:

$$\Phi(t) = e^{At} = \begin{pmatrix} e^{-\alpha t} & te^{-\alpha t} & 0 & 0 \\ 0 & e^{-\alpha t} & 0 & 0 \\ 0 & 0 & \cos(\beta t) & \sin(\beta t) \\ 0 & 0 & -\sin(\beta t) & \cos(\beta t) \end{pmatrix}$$

with its inverse

$$\Phi^{-1}(t) = e^{-At} = \begin{pmatrix} e^{\alpha t} & -te^{\alpha t} & 0 & 0 \\ 0 & e^{\alpha t} & 0 & 0 \\ 0 & 0 & \cos(\beta t) & -\sin(\beta t) \\ 0 & 0 & \sin(\beta t) & \cos(\beta t) \end{pmatrix}.$$

The general solution satisfies:

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + e^{At}\int_0^t e^{-As}g(s)ds,$$

so the particular solution is given by:

$$\begin{aligned} \mathbf{x}_p(t) &= e^{At}\int_0^t e^{-As}g(s)ds, \\ &= e^{At}\int_0^t \begin{pmatrix} e^{\alpha s} & -se^{\alpha s} & 0 & 0 \\ 0 & e^{\alpha s} & 0 & 0 \\ 0 & 0 & \cos(\beta s) & -\sin(\beta s) \\ 0 & 0 & \sin(\beta s) & \cos(\beta s) \end{pmatrix} \begin{pmatrix} e^{-\gamma s} \\ 0 \\ 0 \\ \sin(\omega s) \end{pmatrix} ds \\ &= e^{At}\int_0^t \begin{pmatrix} e^{(\alpha-\gamma)s} \\ 0 \\ -\sin(\beta s)\sin(\omega s) \\ \cos(\beta s)\sin(\omega s) \end{pmatrix} ds. \end{aligned}$$

We examine the **3** integrals above (with Maple), considering the special cases where $\alpha = \gamma$, and $\beta = \omega$.

$$\int_0^t e^{(\alpha-\gamma)s}ds = \begin{cases} \frac{1}{\alpha-\gamma}\left(e^{(\alpha-\gamma)t} - 1\right), & \alpha \neq \gamma, \\ t, & \alpha = \gamma, \end{cases}$$

$$-\int_0^t \sin(\beta s)\sin(\omega s)ds = \begin{cases} \frac{1}{2(\beta^2-\omega^2)}\Big((\beta-\omega)\sin((\beta+\omega)t) - (\beta+\omega)\sin((\beta-\omega)t)\Big), & \beta \neq \omega, \\ \frac{\sin(2\beta t)}{4\beta} - \frac{t}{2} & \beta = \omega, \end{cases}$$

$$\int_0^t \cos(\beta s)\sin(\omega s)ds = \begin{cases} \frac{1}{2(\beta^2-\omega^2)}\Big((\beta+\omega)\cos((\beta-\omega)t) - (\beta-\omega)\cos((\beta+\omega)t) - 2\omega\Big), & \beta \neq \omega, \\ \frac{\sin^2(\beta t)}{2\beta} & \beta = \omega. \end{cases}$$

Technically, there are 4 cases, but since the system is decoupled, the solution will look at the generic case when $\alpha \neq \gamma$ and $\beta \neq \omega$, then combine the cases where $\alpha = \gamma$ and $\beta = \omega$ and understand there are permutations of these solutions. First we write the generic case when $\alpha \neq \gamma$ and $\beta \neq \omega$. The solution of the IVP is

$$\mathbf{x}(t) = \begin{pmatrix} (1+2t)e^{-\alpha t} + \frac{\left(e^{-\gamma t} - e^{-\alpha t}\right)}{\alpha - \gamma} \\ 2e^{-\alpha t} \\ 4\cos(\beta t) - 2\sin(\beta t) + \frac{\cos(\beta t)}{2}\left(\frac{\sin((\beta+\omega)t)}{\beta+\omega} - \frac{\sin((\beta-\omega)t)}{\beta-\omega}\right) + \frac{\sin(\beta t)}{2}\left(\frac{\cos((\beta-\omega)t)}{\beta-\omega} - \frac{\cos((\beta+\omega)t)}{\beta+\omega}\right) \\ -4\sin(\beta t) - 2\cos(\beta t) - \frac{\sin(\beta t)}{2}\left(\frac{\sin((\beta+\omega)t)}{\beta+\omega} - \frac{\sin((\beta-\omega)t)}{\beta-\omega}\right) + \frac{\cos(\beta t)}{2}\left(\frac{\cos((\beta-\omega)t)}{\beta-\omega} - \frac{\cos((\beta+\omega)t)}{\beta+\omega}\right) \end{pmatrix}.$$

With Maple, this is simplified:

$$\mathbf{x}(t) = \begin{pmatrix} (1+2t)e^{-\alpha t} + \frac{\left(e^{-\gamma t} - e^{-\alpha t}\right)}{\alpha - \gamma} \\ 2e^{-\alpha t} \\ 4\cos(\beta t) - 2\sin(\beta t) + \frac{\beta\sin(\omega t) - \omega\sin(\beta t)}{\beta^2 - \omega^2} \\ -4\sin(\beta t) - 2\cos(\beta t) + \frac{\omega\cos(\omega t) - \omega\cos(\beta t)}{\beta^2 - \omega^2} \end{pmatrix}.$$

We combine the two special cases, $\alpha = \gamma$ and $\beta = \omega$, where the first two elements are for $\alpha = \gamma$ and the last two rows are for $\beta = \omega$. The resulting solution to the IVP is

$$\mathbf{x}(t) = \begin{pmatrix} (1+3t)e^{-\alpha t} \\ 2e^{-\alpha t} \\ 4\cos(\beta t) - 2\sin(\beta t) + \frac{\cos(\beta t)\sin(2\beta t)}{4\beta} - \frac{t\cos(\beta t)}{2} + \frac{\sin^3(\beta t)}{2\beta} \\ -4\sin(\beta t) - 2\cos(\beta t) - \frac{\sin(\beta t)\sin(2\beta t)}{4\beta} + \frac{t\sin(\beta t)}{2} + \frac{\cos(\beta t)\sin^2(\beta t)}{2\beta} \end{pmatrix}.$$

With Maple, this is simplified:

$$\mathbf{x}(t) = \begin{pmatrix} (1+3t)e^{-\alpha t} \\ 2e^{-\alpha t} \\ 4\cos(\beta t) - 2\sin(\beta t) + \frac{\sin(\beta t) - \beta t\cos(\beta t)}{2\beta} \\ -4\sin(\beta t) - 2\cos(\beta t) + \frac{t\sin(\beta t)}{2} \end{pmatrix}.$$

It is clear that when $\beta = \omega$, the solution becomes unbounded from the resonance (t term).

b. The homogeneous part of the non-constant, nonhomogeneous system of linear ODEs with $t > 0$:

$$\dot{\mathbf{y}} = \begin{pmatrix} 0 & 1 \\ \frac{3}{t^2} & \frac{1}{t} \end{pmatrix}\mathbf{y} + \begin{pmatrix} -16t^2 \\ 8t \end{pmatrix}, \qquad \mathbf{y}(1) = \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

can be readily seen to satisfy the Cauchy-Euler equation:

$$\ddot{y} - \tfrac{1}{t}\dot{y} - \tfrac{3}{t^2}y = 0,$$

which has the auxiliary equation, $r(r-1) - r - 3 = r^2 - 2r - 3 = (r+1)(r-3) = 0$. This gives the homogeneous solution:

$$y_h(t) = y_1(t) = c_1 t^{-1} + c_2 t^3.$$

With $y_2 = \dot{y}_1$, we can write the fundamental solution and its inverse:

$$\mathbf{\Phi}(t) = \begin{pmatrix} t^{-1} & t^3 \\ -t^{-2} & 3t^2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{\Phi}^{-1}(t) = \begin{pmatrix} \frac{3t}{4} & -\frac{t^2}{4} \\ \frac{1}{4t^3} & \frac{1}{4t^2} \end{pmatrix},$$

using $\det|\mathbf{\Phi}| = 4t$. The variation of parameters method can be used to find the solution:

$$\begin{aligned}
\mathbf{y}(t) &= \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(1)\mathbf{y}(1) + \mathbf{\Phi}(t)\int_1^t \mathbf{\Phi}^{-1}(s)g(s)ds, \\
&= \begin{pmatrix} \frac{1}{t} & t^3 \\ -\frac{1}{t^2} & 3t^2 \end{pmatrix} \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 4 \\ -2 \end{pmatrix} + \begin{pmatrix} \frac{1}{t} & t^3 \\ -\frac{1}{t^2} & 3t^2 \end{pmatrix} \int_1^t \begin{pmatrix} \frac{3s}{4} & -\frac{s^2}{4} \\ \frac{1}{4s^3} & \frac{1}{4s^2} \end{pmatrix} \begin{pmatrix} -16s^2 \\ 8s \end{pmatrix} ds, \\
&= \begin{pmatrix} \frac{1}{t} & t^3 \\ -\frac{1}{t^2} & 3t^2 \end{pmatrix} \begin{pmatrix} \frac{7}{2} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{t} & t^3 \\ -\frac{1}{t^2} & 3t^2 \end{pmatrix} \int_1^t \begin{pmatrix} -14s^3 \\ -\frac{2}{s} \end{pmatrix} ds \\
&= \begin{pmatrix} \frac{7}{2t} + \frac{t^3}{2} \\ -\frac{7}{2t^2} + \frac{3t^2}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{t} & t^3 \\ -\frac{1}{t^2} & 3t^2 \end{pmatrix} \begin{pmatrix} -\frac{7s^4}{2} \\ -2\ln(s) \end{pmatrix} \Bigg|_1^t \\
&= \begin{pmatrix} \frac{7}{2t} + \frac{t^3}{2} \\ -\frac{7}{2t^2} + \frac{3t^2}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{t} & t^3 \\ -\frac{1}{t^2} & 3t^2 \end{pmatrix} \begin{pmatrix} -\frac{7}{2}(t^4 - 1) \\ -2\ln(t) \end{pmatrix} \\
&= \begin{pmatrix} \frac{7}{t} - 3t^3 - 2t^3\ln(t) \\ 5t^2 - \frac{7}{t^2} - 6t^2\ln(t) \end{pmatrix}
\end{aligned}$$