Math 537 - Ordinary Differential Equations

Lecture Notes – Singular Perturbations

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Outline

Algebraic Equation 1

Quadratic Equation: Consider the equation:

$$\varepsilon x^2 + 2x + 1 = 0, \qquad 0 < \varepsilon \ll 1.$$

The ***unperturbed equation*** satisfies:

$$2x + 1 = 0,$$

which has the solution $x = -\frac{1}{2}$.

Note that the unperturbed equation is ***linear***, while the original problem is ***quadratic***.

Consider a *regular perturbation* for solving the original ***quadratic problem***:

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

This inserted into the original ***quadratic equation*** gives:

$$\varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^2 + 2(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) + 1 = 0.$$

Algebraic Equation 2

Regular Perturbation: The ε powers give the following sequence of equations:

$$\begin{aligned}
\varepsilon^0 &: \quad 2x_0 + 1 = 0, \\
\varepsilon^1 &: \quad x_0^2 + 2x_1 = 0, \\
\varepsilon^2 &: \quad 2x_1 x_0 + 2x_2 = 0, \quad \dots
\end{aligned}$$

Sequentially solving these equations give:

$$x_0 = -\tfrac{1}{2}, \qquad x_1 = -\tfrac{1}{8}, \qquad x_2 = -\tfrac{1}{16},$$

which results in the *regular perturbation solution*:

$$x = -\tfrac{1}{2} - \tfrac{1}{8}\varepsilon - \tfrac{1}{16}\varepsilon^2 - \dots$$

For the case $\varepsilon = 0.1$, we solve $0.1x^2 + 2x + 1 = 0$, which has the solutions:

$$x = \frac{-1 \pm \sqrt{0.9}}{0.1} = -0.513167, -19.48683.$$

The approximation gives

$$x = -0.513125.$$

Algebraic Equation

What happened to the other solution of the quadratic?

The *regular perturbation* assumes a leading term of ***order unity***, so this method only recovers a ***root of order unity***.

In this example, the first root gives εx^2 small compared to $2x$ and 1, so it may be ignored.

The second root could be a different order, either large or small.

For the case $\varepsilon = 0.01$, we solve $0.01x^2 + 2x + 1 = 0$, which has the solutions:

$$x = \frac{-1 \pm \sqrt{0.99}}{0.01} = -0.50125629, -199.4987437.$$

The approximation gives $x = -0.50125625$, and we observe the second solution is large.

When εx^2 is not small for a large second root, then either

- εx^2 and 1 are the same order and $2x \ll 1$, or
- εx^2 and $2x$ are the same order and large compared to 1.
- This is an example of ***dominant balancing***, finding which terms are dominant and similar in order.

Algebraic Equation

If εx^2 and 1 are the same order, then $x = \mathcal{O}\left(1/\sqrt{\varepsilon}\right)$ and $2x \ll 1$ doesn't hold.

Thus, εx^2 and $2x$ are the same order with $x = \mathcal{O}\left(1/\varepsilon\right)$, and both dwarf 1, providing a clue to the new scaling to recover the second root.

Choose a new variable y of order unity defined by

$$y = \frac{x}{1/\varepsilon} = \varepsilon x.$$

Inserted into the *original quadratic equation* gives:

$$y^2 + 2y + \varepsilon = 0.$$

A *regular perturbation* uses:

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \dots,$$

so

$$(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \dots)^2 + 2(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \dots) + \varepsilon = 0.$$

Algebraic Equation

From

$$(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \dots)^2 + 2(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \dots) + \varepsilon = 0,$$

the ε powers give the following sequence of equations:

$$\begin{aligned} \varepsilon^0 &: & y_0^2 + 2y_0 &= 0, \\ \varepsilon^1 &: & 2y_0y_1 + 2y_1 + 1 &= 0, \\ \varepsilon^2 &: & 2y_0y_2 + y_1^2 + 2y_2 &= 0, \quad \dots \end{aligned}$$

Sequentially solving these equations give:

$$y_0 = -2, \qquad y_1 = \tfrac{1}{2}, \qquad y_2 = \tfrac{1}{8},$$

so

$$y = -2 + \tfrac{1}{2}\varepsilon + \tfrac{1}{8}\varepsilon^2 \dots,$$

or

$$x = -\tfrac{2}{\varepsilon} + \tfrac{1}{2} + \tfrac{1}{8}\varepsilon \dots$$

For $\varepsilon = 0.1$ and 0.01, we obtain second root approximations of

$$x_2 = -19.4875 \quad \text{and} \quad -199.49875 \qquad (x_{2e} = -19.48683298, \quad -199.4987437).$$

Complex Algebraic Equation

Complex Algebraic Equation: If z_0 is a fixed complex number, then its n^{th} roots are found by solving

$$z^n = z_0.$$

The *fundamental theorem of algebra* states that the're n roots to this equation.

Let $z = re^{i\theta}$ and $z_0 = r_0e^{i\theta_0}$ are complex numbers in ***polar form***.

It follows that $r = r_0^{1/n}$ and $in\theta = i\theta_0 + 2k\pi i$ for $k = 0, \pm1, \pm2, \dots$, so the n roots of z_0 are

$$z = z_0^{1/n} = r_0^{1/n} e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)}, \qquad k = 0, 1, \dots, n-1.$$

If $z_0 = 1$, then this produces the *n roots of unity*.

Example: Find a leading order approximation for the four roots of

$$\varepsilon x^4 - x - 1 = 0, \qquad \text{with} \quad 0 < \varepsilon \ll 1.$$

When $\varepsilon = 0$, this only has the ***single root***, $x = -1$, which is order 1.

Dominant balancing is used to find the *leading order other roots*.

Complex Algebraic Equation

Example: If the first and third terms of

$$\varepsilon x^4 - x - 1 = 0,$$

balance, then $x = \mathcal{O}\left(\varepsilon^{-1/4}\right)$, which is large, so is ***inconsistent***.

If the first and second terms ***balance***, then $x = \mathcal{O}\left(\varepsilon^{-1/3}\right)$, which is large compared to **1**.

This suggests re-scaling with $y = \varepsilon^{1/3}x$, which gives

$$y^4 - y - \varepsilon^{1/3} = 0.$$

The leading order becomes $y^4 - y = 0$, which after discarding $y = 0$ gives $y = 1, \quad e^{2\pi i/3}, \quad e^{-2\pi i/3}$.

It follows that the leading order ***four roots*** are

$$x = -1, \quad \varepsilon^{-1/3}, \quad \varepsilon^{-1/3}e^{2\pi i/3}, \quad \varepsilon^{-1/3}e^{-2\pi i/3}.$$

The last three are large.

Complex Algebraic Equation

Example: The leading order ***four roots*** are:

$$x = -1, \quad \varepsilon^{-1/3}, \quad \varepsilon^{-1/3}e^{2\pi i/3}, \quad \varepsilon^{-1/3}e^{-2\pi i/3}.$$

which for $\varepsilon = 0.001$ gives:

$$x = -1, \quad 10, \quad -5 \pm i5\sqrt{3} = -5 \pm 8.660254i.$$

Maple gives the **4 roots** of $0.001x^4 - x - 1 = 0$, as

$$x = -0.99900398, \quad 10.313290, \quad -4.6571430 \pm 8.6815875i.$$

Higher-order approximations are found with the series:

$$y = y_0 + \varepsilon^{1/3}y_1 + \varepsilon^{2/3}y_2 + \dots,$$

so

$$(y_0 + \varepsilon^{1/3}y_1 + \dots)^4 - (y_0 + \varepsilon^{1/3}y_1 + \dots) - \varepsilon^{1/3} = 0, \qquad \text{or}$$

$$\begin{aligned} \varepsilon^0 &: \quad y_0^4 - y_0 = 0, \\ \varepsilon^{1/3} &: \quad 4y_0^3y_1 - y_1 - 1 = 0. \end{aligned}$$

Complex Algebraic Equation

Example: For $y = y_0 + \varepsilon^{1/3}y_1 + \dots$, the ε^0-order terms gave the leading order approximation, and specifically gave $y_0^3 = 1$.

Since the $\varepsilon^{1/3}$-order term satisfies the equation, $4y_0^3y_1 - y_1 - 1 = 0$, it follows that:

$$y_1 = \frac{1}{4y_0^3 - 1} \qquad \text{or} \qquad y_1 = \tfrac{1}{3}.$$

Thus, the two term approximation is

$$y \approx y_0 + \tfrac{1}{3}\varepsilon^{1/3} \qquad \text{or} \qquad x \approx \varepsilon^{-1/3}y_0 + \tfrac{1}{3}.$$

With $\varepsilon = 0.001$ from above, this improves our approximations to

$$x \approx x = -1, \quad 10.33333, \quad -4.66667 \pm 8.660254i,$$

which compare quite favorably to the **Maple** solution:

$$x = -0.99900398, \quad 10.313290, \quad -4.6571430 \pm 8.6815875i.$$

Boundary Value Problem

Consider the **boundary value problem (BVP)**:

$$y'' - y = 0, \qquad y(0) = A, \quad \text{and} \quad y(1) = B,$$

which again has the general solution $y(t) = c_1e^t + c_2e^{-t}$.

With algebra, the ***unique solution*** becomes

$$y(t) = -\frac{(A\mathrm{e} - B)\,\mathrm{e}^{-t}}{\mathrm{e}^{-1} - \mathrm{e}} + \frac{\left(A\mathrm{e}^{-1} - B\right)\mathrm{e}^{t}}{\mathrm{e}^{-1} - \mathrm{e}}$$

Since $\sinh(t)$ and $\sinh(1-t)$ are linearly independent combinations of e^t and e^{-t}, we could write

$$y(t) = d_1\sinh(t) + d_2\sinh(1-t).$$

The algebra makes it much easier to see that

$$y(t) = \frac{B}{\sinh(1)}\sinh(t) + \frac{A}{\sinh(1)}\sinh(1-t).$$

Harmonic Oscillator 1

Example (Harmonic Oscillator): Consider the BVP:

$$y'' + y = 0, \qquad y(0) = A, \quad y(1) = B,$$

which has the general solution

$$y(t) = c_1 \cos(t) + c_2 \sin(t).$$

The boundary conditions are easily solved to give

$$y(t) = A\cos(t) + \frac{B - A\cos(1)}{\sin(1)}\sin(t).$$

This again gives a ***unique solution***, but the denominator of $\sin(1)$ suggests potential problems at certain t values.

Harmonic Oscillator 2

Example (Harmonic Oscillator): Now consider the BVP:

$$y'' + y = 0, \qquad y(0) = A, \quad y(\pi) = B,$$

which again has the general solution

$$y(t) = c_1 \cos(t) + c_2 \sin(t).$$

The condition $y(0) = A$ implies $c_1 = A$. However, $y(\pi) = B$ gives

$$y(\pi) = A\cos(\pi) + c_2\sin(\pi) = -A = B.$$

This only has a solution if $B = -A$. Furthermore, if $B = -A$, the arbitrary constant c_2 remains undetermined, so takes any value.

- If $B \neq -A$, then **no solution exists**.
- If $B = -A$, then **infinity many solutions exist** and satisfy

$$y(t) = A\cos(t) + c_2\sin(t), \quad \text{where } c_2 \text{ is arbitrary.}$$

General Case

Theorem (Boundary Value Problem)

Consider the second order linear BVP

$$y'' + py' + qy = 0, \qquad y(a) = A, \quad y(b) = B,$$

where p, q, $a \neq b$, A, and B are constants. Exactly one of the following conditions hold:

- *There is a **unique solution** to the BVP.*
- *There is **no solution** to the BVP.*
- *There are **infinity many solutions** to the BVP.*

Boundary value problems have many practical applications and form a base for many problems in partial differential equations.

Boundary Layers 1

Boundary Layers: Consider the ***BVP***:

$$\begin{aligned} \varepsilon y'' + (1+\varepsilon)y' + y &= 0, \qquad && 0 < x < 1, \quad 0 < \varepsilon \ll 1 \\ y(0) &= 0, && y(1) = 1. \end{aligned} \tag{1}$$

Begin by solving this equation, and examining its behavior as ε varies.

The ***characteristic equation*** satisfies:

$$\varepsilon\lambda^2 + (1+\varepsilon)\lambda + 1 = (\lambda + 1)(\varepsilon\lambda + 1) = 0,$$

which gives $\lambda_1 = -1$ and $\lambda_2 = -\frac{1}{\varepsilon}$.

The general solution of (1) is

$$y(x) = c_1 e^{-x} + c_2 e^{-x/\varepsilon}.$$

the boundary conditions give:

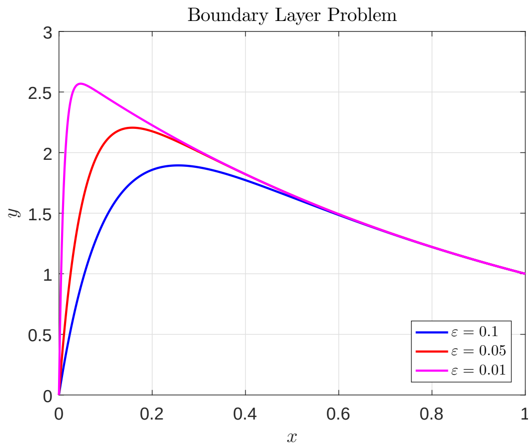
$$y(0) = c_1 + c_2 = 0 \quad \text{or} \quad c_2 = -c_1, \qquad \text{and} \qquad y(1) = c_1\left(e^{-1} - e^{-1/\varepsilon}\right) = 1.$$

The ***unique solution*** to the ***BVP*** (1) is

$$y(x) = \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}.$$

Boundary Layers

Below are graphs of solutions for several values of ε to the ***BVP***. The graph shows early rapid rise followed by slow decay (common in drug kinetic problems).



Boundary Layers

Perturbation Method: Attempt to naively solve the ***BVP***

$$\begin{aligned} \varepsilon y'' + (1+\varepsilon)y' + y = 0, \qquad & 0 < x < 1, \quad 0 < \varepsilon \ll 1 \\ y(0) = 0, \qquad & y(1) = 1, \end{aligned}$$

with our ***regular perturbation method***.

Assume a solution

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots$$

The differential equation becomes:

$$\begin{aligned} \varepsilon(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \dots) + (y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \dots) & \\ + \varepsilon(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \dots) + (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots) &= 0. \end{aligned}$$

Equating coefficients of like powers of ε gives the sequences of problems:

$$\begin{aligned} y_0' + y_0 &= 0, \\ y_1' + y_1 &= -y_0'' - y_0', \dots \end{aligned}$$

Boundary Layers

Perturbation Method: From before the leading order problem is

$$y_0' + y_0 = 0, \qquad y_0(0) = 0, \quad y_0(1) = 1,$$

which is readily seen to have a problem as it is a **first order ODE** with two conditions.

The general solution is:

$$y_0(x) = ce^{-x}.$$

If $y_0(0) = 0$, then the solution satisfies $c = 0$ or $y(x) \equiv 0$, which cannot work.

If $y_0(1) = 1$, then the solution becomes

$$y(x) = e^{1-x},$$

which fails at $x = 0$.

It follows that at the first step of the ***regular perturbation method*** the method fails.

Boundary Layers

Boundary Layers: Recall the ***unique solution*** to the ***BVP*** (1) is

$$y(x) = \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}.$$

For small ε the graph showed $y(x)$ rapidly increasing in a small interval near the origin, which is called a ***boundary layer***.

In the interval outside this region the solution slowly decayed, and this region is called the ***outer layer***.

This suggests the need for **two spatial scales**.

With the solution for this problem, we examine each term in the ***ODE***.

$$\begin{aligned} y'(x) &= \frac{1}{e^{-1} - e^{-1/\varepsilon}} \left(-e^{-x} + \tfrac{1}{\varepsilon} e^{-x/\varepsilon} \right), \\ y''(x) &= \frac{1}{e^{-1} - e^{-1/\varepsilon}} \left(e^{-x} - \tfrac{1}{\varepsilon^2} e^{-x/\varepsilon} \right). \end{aligned}$$

Boundary Layers 6

Boundary Layers: Near the origin, say $x = \varepsilon$, we evaluate both y' and y'', then it is an easy calculation to see that

$$y'(\varepsilon) = \mathcal{O}\left(\varepsilon^{-1}\right) \qquad \text{and} \qquad y''(\varepsilon) = \mathcal{O}\left(\varepsilon^{-2}\right).$$

It follows that these terms in the original ***ODE*** are not small for x small, so requires a rescaling.

For larger x, $\varepsilon y''(x) = \mathcal{O}(\varepsilon)$ and similar for $\varepsilon y'(x)$, so these terms may be ignored, so the original *regular perturbation method* should provide a very good *outer approximation*,

$$y_0(x) = e^{1-x}.$$

For ε small, $e^{-1} - e^{-1/\varepsilon} \approx e^{-1}$, so expect an *inner approximation* given by:

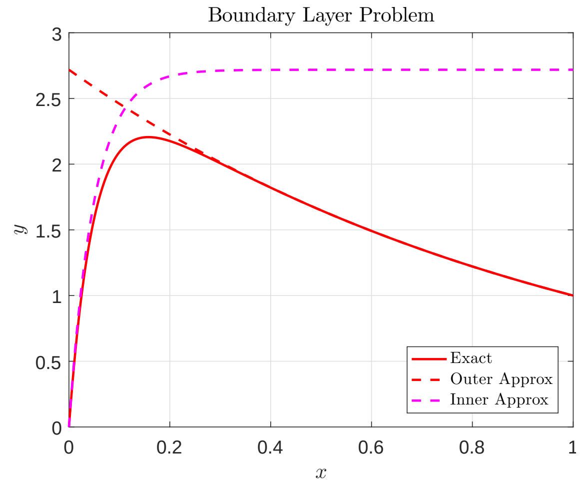
$$y_i(x) = e - e^{1-x/\varepsilon},$$

based on the known solution.

Here we knew the exact solution, so were able to obtain the *inner approximation*. We need to develop a scaling technique to find this *boundary layer approximation*.

Boundary Layers 7

The graph below shows the exact solution of the original ***ODE*** with overlays for the *outer approximation* and *inner approximation*.



Inner and Outer Approximations 1

Inner and Outer Approximations: We return to the ***BVP***:

$$\begin{aligned} \varepsilon y'' + (1+\varepsilon)y' + y &= 0, \qquad & 0 < x < 1, \quad 0 < \varepsilon \ll 1 \\ y(0) &= 0, & y(1) = 1. \end{aligned}$$

We showed that the original *regular perturbation method* provided a very good *outer approximation*,

$$y_0(x) = e^{1-x},$$

by setting $\varepsilon = 0$ and selecting only the *boundary condition*, $y(1) = 1$.

There are significant changes in the *boundary layer*, which suggests making a length scale on the order of a function of ε, $\delta(\varepsilon)$.

Consider the change of variables:

$$\xi = \frac{x}{\delta(\varepsilon)} \qquad \text{and} \qquad Y(\xi) = y(\delta(\varepsilon)\xi).$$

With the chain rule the ***ODE*** becomes:

$$\frac{\varepsilon}{\delta(\varepsilon)^2} Y''(\xi) + \frac{(1+\varepsilon)}{\delta(\varepsilon)} Y'(\xi) + Y(\xi) = 0,$$

where prime is differentiation with respect to ξ.

Inner and Outer Approximations 2

Inner and Outer Approximations: The ***ODE***

$$\frac{\varepsilon}{\delta(\varepsilon)^2} Y''(\xi) + \frac{(1+\varepsilon)}{\delta(\varepsilon)} Y'(\xi) + Y(\xi) = 0,$$

has the coefficients $\frac{\varepsilon}{\delta(\varepsilon)^2}$, $\frac{1}{\delta(\varepsilon)}$, $\frac{\varepsilon}{\delta(\varepsilon)}$, and 1.

We know the first coefficient must be significant. The possibilities are:

1. The terms $\varepsilon/\delta(\varepsilon)^2$ and $1/\delta(\varepsilon)$ have the same order, while $\varepsilon/\delta(\varepsilon)$ and 1 are comparatively small.
2. The terms $\varepsilon/\delta(\varepsilon)^2$ and 1 have the same order, while $1/\delta(\varepsilon)$ and $\varepsilon/\delta(\varepsilon)$ are comparatively small.
3. The terms $\varepsilon/\delta(\varepsilon)^2$ and $\varepsilon/\delta(\varepsilon)$ have the same order, while $1/\delta(\varepsilon)$ and 1 are comparatively small.

Only Case (1) is possible.

For Case (2) if $\varepsilon/\delta(\varepsilon)^2 \sim 1$, then $\delta(\varepsilon) = \mathcal{O}\left(\sqrt{\varepsilon}\right)$ and $1/\delta(\varepsilon)$ is not small compared to 1.

For Case (3) if $\varepsilon/\delta(\varepsilon)^2 \sim \varepsilon/\delta(\varepsilon)$, then $\delta(\varepsilon) = \mathcal{O}(1)$ and leads to the *outer approximation*.

Inner and Outer Approximations 3

Inner and Outer Approximations: For Case (1) if $\varepsilon/\delta(\varepsilon)^2 \sim 1/\delta(\varepsilon)$, then $\delta(\varepsilon) = \mathcal{O}(\varepsilon)$, so take

$$\delta(\varepsilon) = \varepsilon.$$

This leads to the scaled ***ODE***

$$Y'' + Y' + \varepsilon Y' + \varepsilon Y = 0,$$

which is amenable to *regular perturbation*.

The *leading-order approximation* ($\varepsilon = 0$) gives:

$$Y'' + Y' = 0, \qquad Y(0) = 0.$$

The solution to this ***initial value problem*** is

$$Y(x) = c_0 \left(1 - e^{-\xi}\right),$$

which is the ***inner approximation*** for $x = \mathcal{O}(\varepsilon)$.

Matching 1

Matching Approximations: The ***BVP***:

$$\begin{aligned} \varepsilon y'' + (1+\varepsilon)y' + y &= 0, && 0 < x < 1, \quad 0 < \varepsilon \ll 1 \\ y(0) &= 0, && y(1) = 1, \end{aligned}$$

gave the ***inner approximation***, y_i and ***outer approximation***, y_0:

$$\begin{aligned} y_0(x) &= e^{1-x}, && x = \mathcal{O}(1), \\ y_i(x) &= c_0\left(1 - e^{-x/\varepsilon}\right), && x = \mathcal{O}(\varepsilon), \end{aligned}$$

for the appropriate range of x.

There is still an arbitrary constant, c_0.

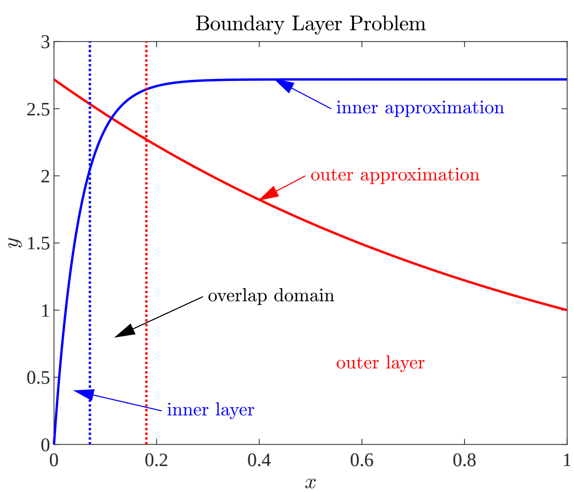
The goal is to construct a single composite expansion in ε that is uniformly valid for $x \in [0,1]$, as $\varepsilon \to 0$.

The width of the ***boundary layer*** varies according to the scaling factor $\delta(\varepsilon)$, so it is reasonable to have the ***inner*** and ***outer expansions*** agree to some order in an overlap domain.

Matching 2

Matching Approximations: The ***inner approximation***, y_i, is valid for $x = \mathcal{O}(\varepsilon)$, while the ***outer approximation***, y_0, is valid for $x = \mathcal{O}(1)$.

This suggests an ***overlap region***, which is characterized by $x = \mathcal{O}(\sqrt{\varepsilon})$.



Matching 3

Matching Approximations: The ***overlap region*** suggests creating an intermediate variable, which is $\mathcal{O}(\sqrt{\varepsilon})$, say

$$\eta = \frac{x}{\sqrt{\varepsilon}}.$$

The ***inner approximation***, y_i, in terms of the intermediate variable, should agree with the ***outer approximation***, y_0, in the limit as $\varepsilon \to 0$ or for fixed η

$$\lim_{\varepsilon\to 0+} y_0(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon\to 0+} y_i(\sqrt{\varepsilon}\eta).$$

For this example

$$\lim_{\varepsilon\to 0+} y_0(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon\to 0+} e^{1-\sqrt{\varepsilon}\eta} = e,$$

and

$$\lim_{\varepsilon\to 0+} y_i(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon\to 0+} c_0\left(1 - e^{-\eta/\sqrt{\varepsilon}}\right) = c_0.$$

Thus, matching requires that $c_0 = e$, and the ***inner approximation*** becomes:

$$y_i(x) = e\left(1 - e^{-x/\varepsilon}\right).$$

Matching 4

Matching Approximations: Because our approximations only use leading order terms, the introduction of an *intermediate variable* is not necessary.

The *matching condition* simply requires:

$$\lim_{x\to 0+} y_0(x) = \lim_{\xi\to\infty} Y_i(\xi) = e,$$

which is stating that the *outer approximation*, as the *outer variable* moves into the *inner region*, must equal the *inner approximation*, as the *inner variable* moves to the *outer region*.

Higher order approximations require more complex **matching schemes**.

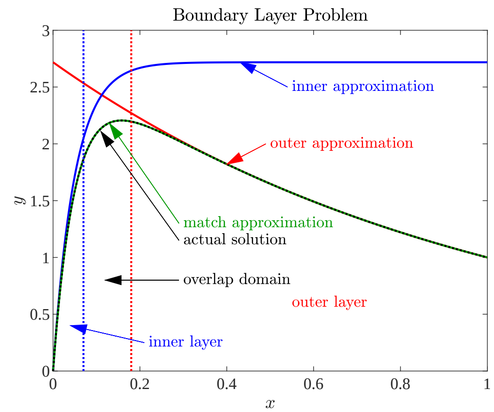
Uniform Approximations: To obtain a uniformly valid approximation for $x \in [0,1]$, examine the sum of the *inner* and *outer approximations*:

$$\begin{aligned} y_0(x) + y_i(x) &= e^{1-x} + e - e^{1-x/\varepsilon}, \\ &= \begin{cases} e^{1-x} + e, & x = \mathcal{O}(1), \\ 2e - e^{1-x/\varepsilon}, & x = \mathcal{O}(\varepsilon). \end{cases} \end{aligned}$$

Matching 5

Matching Approximations: From the composite expansion the common limit (e) is subtracted to obtain a *uniform approximation*, which follows the *inner approximation* for $x = \mathcal{O}(\varepsilon)$, the *outer approximation* for $x = \mathcal{O}(1)$, and *matches uniformly* for $x = \mathcal{O}\left(\sqrt{\varepsilon}\right)$:

$$y_u(x) = y_0(x) + y_i(x) - e = e^{1-x} - e^{1-x/\varepsilon}.$$



BVP Example 2 1

BVP Example 2: Consider the *BVP*:

$$\begin{aligned} \varepsilon y'' + y' &= 2x, & 0 < x < 1, \quad 0 < \varepsilon \ll 1 \\ y(0) &= 1, & y(1) = 1. \end{aligned} \tag{2}$$

Begin by solving this equation.

The *characteristic equation* satisfies $\varepsilon\lambda^2 + \lambda = \lambda(\varepsilon\lambda + 1) = 0$, so the *homogeneous solution* of (2) is

$$y_h(x) = c_1 + c_2 e^{-x/\varepsilon}.$$

The *particular solution* is easily seen to satisfy:

$$y_p(x) = x^2 - 2\varepsilon x.$$

With the boundary conditions the *unique solution* becomes

$$y(x) = \frac{(2\varepsilon - 1)e^{-\frac{x}{\varepsilon}} + e^{-\frac{1}{\varepsilon}} - 2\varepsilon}{e^{-\frac{1}{\varepsilon}} - 1} + x^2 - 2\varepsilon x.$$

BVP Example 2 2

The *Regular Perturbation Method* allows obtaining the *outer solution* for $x = \mathcal{O}(1)$. This is accomplished by letting $\varepsilon = 0$ in (2) and taking only the outer *boundary condition*, so

$$y' = 2x, \qquad \text{with} \quad y(1) = 1.$$

This is easily solved giving the *outer solution*:

$$y_o(x) = x^2.$$

The next step is to find the appropriate scaling for the *inner solution* by letting

$$\xi = \frac{x}{\delta(\varepsilon)}, \qquad \text{and taking} \qquad Y(\xi) = y(x).$$

The original *BVP*, $\varepsilon y'' + y' = 2x$, becomes

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{1}{\delta} Y' = 2\delta\xi.$$

If $\varepsilon/\delta(\varepsilon)^2 \sim 2\delta(\varepsilon)$, then $\delta(\varepsilon) = \mathcal{O}\left(\varepsilon^{1/3}\right)$ and the term with Y' is $\mathcal{O}\left(\varepsilon^{-1/3}\right)$, which is large or **dominant**.

BVP Example 2

It follows that $\varepsilon/\delta(\varepsilon)^2 \sim 1/\delta(\varepsilon)$, so $\delta(\varepsilon) = \mathcal{O}(\varepsilon)$ and we take $\delta(\varepsilon) = \varepsilon$.

The scaled ***BVP*** becomes:

$$Y'' + Y' = 2\varepsilon^2\xi.$$

This has a first order approximation ($Y'' + Y' = 0$):

$$Y_i(\xi) = c_1 + c_2 e^{-\xi}, \qquad \text{with} \qquad Y_i(0) = 1,$$

so

$$Y_i(\xi) = (1 - c_2) + c_2 e^{-\xi} \qquad \text{or} \qquad y_i(x) = (1 - c_2) + c_2 e^{-\frac{x}{\varepsilon}},$$

which gives the ***inner approximation***.

For the ***matching condition***, we introduce an ***overlap region***, $\mathcal{O}\left(\sqrt{\varepsilon}\right)$ by letting $x = \sqrt{\varepsilon}\eta$. The ***matching condition*** becomes:

$$\lim_{\varepsilon \to 0^+} y_o(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon \to 0^+} y_i(\sqrt{\varepsilon}\eta),$$

or

$$\lim_{\varepsilon \to 0^+} \varepsilon\eta^2 = 0 = \lim_{\varepsilon \to 0^+} (1 - c_2) + c_2 e^{-\frac{\eta}{\sqrt{\varepsilon}}} = 1 - c_2.$$

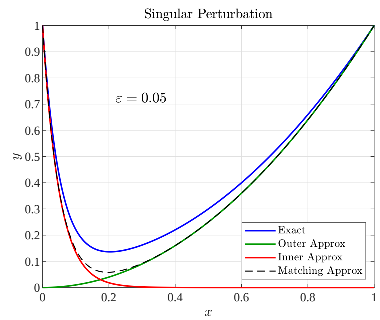
BVP Example 2

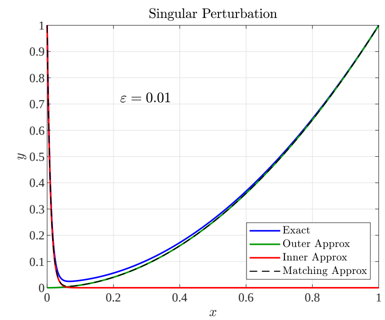
The ***matching condition*** gave $c_2 = 1$, so the ***inner approximation*** and ***outer approximation*** are:

$$y_i(x) = e^{-\frac{x}{\varepsilon}} \qquad \text{and} \qquad y_o(x) = x^2.$$

Since the common limit in the ***overlap region*** is zero, the ***uniform composite approximation*** satisfies:

$$y_u(x) = x^2 + e^{-\frac{x}{\varepsilon}}.$$





Singular Perturbation

Singular Perturbation: Below are some indicators that the ***regular perturbation method*** will fail.

1. When a small parameter multiplies the highest derivative in the problem.
2. When a small parameter in a problem is set to zero results in a fundamentally different problem.
3. When problems occur on infinite domains, like when secular terms arise.
4. When singular points are present in the interval of interest.
5. When the equations that model physical processes have multiple time or spatial scales.

Singular Perturbation

Singular Perturbation: Our examples all had their ***boundary layer*** at $x = 0$.

- ***Boundary layers*** can occur at any point, including the right end point or an interior point, and multiple ***boundary layers*** can occur.
- ***Boundary layers*** can occur in ***initial value problems***.
- Assume the ***boundary layer*** at $x = 0$, then if incorrect the procedure will break down when trying to match inner and outer solutions.
- For a ***boundary layer*** at the right end the inner variable is scaled

$$\xi = \frac{x_0 - x}{\delta(\varepsilon)}.$$

- This case gives

$$\frac{dy}{dx} = -\frac{1}{\delta(\varepsilon)}\frac{dY}{d\xi} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{1}{\delta(\varepsilon)^2}\frac{d^2Y}{d\xi^2}.$$

- The matching condition is

$$\lim_{\xi \to \infty} Y_i(\xi) = \lim_{x \to x_0} y_0(x).$$

- Our examples had a scaling of $\delta(\varepsilon) = \varepsilon$, but this is not the rule in general.
- Refinements for ***higher order approximations*** are needed, and often problems need ***significant modifications***. (Research ongoing.)

General Singular Perturbation

General Singular Perturbation: Linear equations with variable coefficients can be completely characterized.

Theorem (Singular Perturbation)

Consider the ***boundary value problem***

$$\begin{aligned} \varepsilon y'' + p(x)y' + q(x)y = 0, &\qquad 0 < x < 1, \quad 0 < \varepsilon \ll 1, \\ y(0) = a, &\qquad y(1) = b, \end{aligned}$$

where p and q are continuous functions with $p(x) > 0$ for $x \in [0,1]$. Then there exists a boundary layer at $x = 0$ with inner and outer approximations given by

$$\begin{aligned} y_i(x) &= C_1 + (a - C_1)e^{-p(0)x/\varepsilon}, \\ y_o(x) &= b\exp\left(\int_x^1 \frac{q(s)}{p(s)}ds\right), \end{aligned}$$

where

$$C_1 = b\exp\left(\int_0^1 \frac{q(s)}{p(s)}ds\right).$$

General Singular Perturbation

Proof: Assume that the ***boundary value problem*** of the theorem,

$$\varepsilon y'' + p(x)y' + q(x)y = 0,$$

has a ***boundary layer*** at $x = 0$.

It follows that the ***outer approximation*** satisfies the ***initial value problem***:

$$p(x)y_o' + q(x)y_o = 0, \qquad y_o(1) = b.$$

Solving this ***first order linear ODE*** gives:

$$y_o(x) = b\exp\left(\int_x^1 \frac{q(s)}{p(s)}ds\right).$$

This solution is a good approximation for $x = \mathcal{O}(1)$.

General Singular Perturbation

The scaled variable in the ***boundary layer***, $\xi = x/\delta(\varepsilon)$, is introduced, where $\delta(\varepsilon)$ is to be determined.

If $Y(\xi) = y(\delta(\varepsilon)\xi)$, then the ***ODE*** becomes:

$$\frac{\varepsilon}{\delta(\varepsilon)^2}Y'' + \frac{p(\delta(\varepsilon)\xi)}{\delta(\varepsilon)}Y' + q(\delta(\varepsilon)\xi)Y = 0.$$

The coefficients behave like $\frac{\varepsilon}{\delta(\varepsilon)^2}$, $\frac{p(0}{\delta(\varepsilon)}$, and $q(0)$, as $\varepsilon \to 0^+$.

The ***dominant balance*** is $\frac{\varepsilon}{\delta(\varepsilon)^2} \sim \frac{p(0}{\delta(\varepsilon)}$, so $\delta(\varepsilon) = \mathcal{O}(\varepsilon)$.

It suffices to take $\delta(\varepsilon) = \varepsilon$, so the rescaled ***ODE*** becomes:

$$Y'' + p(\varepsilon\xi)Y' + \varepsilon q(\varepsilon\xi)Y = 0,$$

which to a leading order becomes

$$Y_i'' + p(0)Y_i' = 0.$$

General Singular Perturbation

The ***ODE*** for the ***inner approximation*** is

$$Y_i'' + p(0)Y_i' = 0,$$

which has the general solution:

$$Y_i(\xi) = c_1 + c_2e^{-p(0)\xi},$$

which with the other ***boundary condition*** $Y_i(0) = a$ gives

$$y_i(x) = c_1 + (a - c_1)e^{-p(0)x/\varepsilon}.$$

Introduce the intermediate scaling variable $\eta = x/\sqrt{\varepsilon}$, then the ***matching condition*** for fixed η is

$$\lim_{\varepsilon\to0^+} y_i(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon\to0^+} y_o(\sqrt{\varepsilon}\eta),$$

or equivalently

$$\lim_{\varepsilon\to0^+} c_1 + (a - c_1)e^{-p(0)\eta/\sqrt{\varepsilon}} = \lim_{\varepsilon\to0^+} b\exp\left(\int_{\sqrt{\varepsilon}\eta}^1 \frac{q(s)}{p(s)}ds\right).$$

General Singular Perturbation 5

This forces

$$c_1 = b\exp\left(\int_0^1 \frac{q(s)}{p(s)}ds\right).$$

A ***uniform composite approximation*** is given by

$$\begin{aligned} y_u(x) &= y_o(x) + y_i(x) - c_1, \\ &= b\exp\left(\int_x^1 \frac{q(s)}{p(s)}ds\right) + (a - c_1)e^{-p(0)x/\varepsilon}. \end{aligned}$$

It can be shown that $y_u(x) - y(x) = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0^+$, uniformly on $[0,1]$, where $y(x)$ is the exact solution.

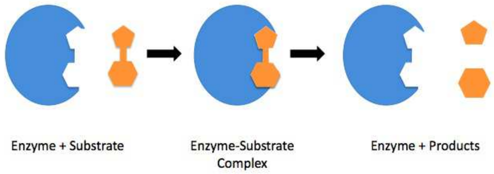
If $p(x) < 0$ for $x \in [0,1]$, then no match is possible because of the exponential growth of $y_i(x)$ (unless $c_1 = a$). However, with $p(x) < 0$ a match is possible for the boundary layer occurring at $x = 1$.

It follows that a ***boundary layer*** occurs at $x = 0$, if $p(x) > 0$, and it occurs at $x = 1$, if $p(x) < 0$. If $p(x)$ changes signs for $x \in [0,1]$, then an ***interior boundary layer*** is possible, and these points are called ***turning point problems***.

Enzyme Kinetics 1

Enzyme Kinetics: Chemical processes are very dependent on concentrations of the chemical species and can readily be described by ***differential equations***.

The chemical reactions are usually nonlinear problems and often occur on different time scales, which make these problems a rich source of singular perturbation problems and other types of analyses.



The enzyme reaction is given by the chemical equation:

$$S + E \underset{k_1}{\overset{k_{-1}}{\rightleftarrows}} C \xrightarrow{k_2} P + E,$$

which says that a molecule of substrate, S, combines with a molecule of enzyme, E, to form a molecule of complex, C, which can either disassociate or proceed forward to produce a product, P.

Enzyme Kinetics 2

Enzyme Kinetics: The enzyme reaction given by:

$$S + E \underset{k_1}{\overset{k_{-1}}{\rightleftarrows}} C \xrightarrow{k_2} P + E$$

can be written as the following system of ***ODEs***:

$$\begin{aligned} \frac{dS}{d\tau} &= -k_1SE + k_{-1}C, \\ \frac{dE}{d\tau} &= -k_1SE + (k_{-1} + k_2)C, \\ \frac{dC}{d\tau} &= k_1SE - (k_{-1} + k_2)C, \\ \frac{dP}{d\tau} &= k_2C, \end{aligned}$$

where E, S, C, and P are concentrations of enzyme, substrate, complex, and product, respectively.

Enzymatic reactions generally form the complex very rapidly with the formation of product being the slowest (rate limiting) reaction.

Initially, it is assumed that $S(0) = S_0$, $E(0) = E_0$, $C(0) = 0$, and $P(0) = 0$, where E_0 is small relative compared to S_0.

Enzyme Kinetics 3

Enzyme Kinetics: The system of ***ODEs*** shows that P is immediately found by integrating the solution for C.

Since there is no loss of material in this system, there are the conservation laws:

$$E + C = E_0 \qquad \text{and} \qquad S + C + P = S_0.$$

This allows reduction to a system of nonlinear equations:

$$\begin{aligned} \frac{dS}{d\tau} &= -k_1E_0S + (k_{-1} + k_1S)C, \\ \frac{dC}{d\tau} &= k_1E_0S - (k_2 + k_{-1} + k_1S)C. \end{aligned}$$

Generally, there is a rapid rise (boundary layer) of the complex, C, followed by a much slower conversion of the substrate, S, into the product P.

This problem is rescaled and solved as a ***singular perturbation problem***.

Scaling 1

Enzyme Kinetics: For the system of nonlinear equations:

$$\begin{aligned}\frac{dS}{d\tau} &= -k_1E_0S + (k_{-1} + k_1S)C,\\ \frac{dC}{d\tau} &= k_1E_0S - (k_2 + k_{-1} + k_1S)C,\end{aligned}$$

we take

$$x = \frac{S}{S_0}, \qquad y = \frac{C}{E_0}, \qquad t = \frac{\tau}{T},$$

where T is still an unknown time scale.

The resulting scaled system is:

$$\begin{aligned}\frac{dx}{dt} &= -k_1E_0Tx + (k_{-1} + k_1S_0x)T\frac{E_0}{S_0}y,\\ \frac{dy}{dt} &= k_1S_0Tx - (k_2 + k_{-1} + k_1S_0x)Ty.\end{aligned}$$

There are two obvious time scales, T:

$$T_s = \frac{1}{k_1E_0} \qquad \text{and} \qquad T_f = \frac{1}{k_1S_0},$$

where the subscripts denote the slow and fast time scales, as typically E_0 is much smaller than S_0.

Scaling 2

Scaled System: For the scaled system:

$$\begin{aligned}\frac{dx}{dt} &= -k_1E_0Tx + (k_{-1} + k_1S_0x)T\frac{E_0}{S_0}y,\\ \frac{dy}{dt} &= k_1S_0Tx - (k_2 + k_{-1} + k_1S_0x)Ty,\end{aligned}$$

with the slow time scale, $T_s = \frac{1}{k_1E_0}$ and the defined parameters:

$$\mu = \frac{k_{-1}}{k_1S_0}, \qquad \lambda = \frac{k_{-1} + k_2}{k_1S_0}, \qquad \varepsilon = \frac{E_0}{S_0},$$

we obtain

$$\begin{aligned}\frac{dx}{dt} &= -x + (\mu + x)y,\\ \varepsilon\frac{dy}{dt} &= x - (\lambda + x)y.\end{aligned}$$

Usually, k_1 and k_{-1} are relatively large (fast equilibrating dynamics) and are sometimes used in what is called ***quasi-steady state analysis*** for a differential equation.

This results in μ and λ being $\mathcal{O}(1)$ with $\varepsilon \ll 1$.

Outer Approximation

Outer Approximation: The scaled system is:

$$\begin{aligned}\frac{dx}{dt} &= -x + (\mu + x)y,\\ \varepsilon\frac{dy}{dt} &= x - (\lambda + x)y.\end{aligned}$$

Let $x = x_0 + \varepsilon x_1 + \mathcal{O}(\varepsilon^2)$ and $y = y_0 + \varepsilon y_1 + \mathcal{O}(\varepsilon^2)$, then the zeroth order approximation is:

$$\begin{aligned}\frac{dx_0}{dt} &= -x_0 + (\mu + x_0)y_0,\\ 0 &= x_0 - (\lambda + x_0)y_0,\end{aligned}$$

where the last equation becomes an ***algebraic equation***, $y_0 = \frac{x_0}{\lambda + x_0}$.

The system reduces to a single ***first order nonlinear ODE***:

$$\frac{dx_0}{dt} = \frac{(\mu - \lambda)x_0}{\lambda + x_0}, \qquad x_0(0) = 1.$$

Separation of variables solves the second equation, giving the implicit solution:

$$x_0(t) + \lambda\ln\big(x_0(t)\big) = (\mu - \lambda)t + c_0,$$

where c_0 is a constant and $x_0(0) = c_0 = 1$.

Inner Approximation 1

Inner Approximation: Now change the time scale for $t = \mathcal{O}(\varepsilon)$ by creating the fast timescale:

$$\bar{t} = \frac{t}{\varepsilon} = \frac{\tau}{T_f}.$$

By letting $X(\bar{t}) = x(\varepsilon\bar{t})$ and $Y(\bar{t}) = y(\varepsilon\bar{t})$, we obtain the scaled system:

$$\begin{aligned}\frac{dX}{dt} &= \varepsilon\big(-X + (\mu + X)Y\big),\\ \frac{dY}{dt} &= X - (\lambda + X)Y.\end{aligned}$$

Let $X = X_0 + \varepsilon X_1 + \mathcal{O}(\varepsilon^2)$ and $Y = Y_0 + \varepsilon Y_1 + \mathcal{O}(\varepsilon^2)$, then the zeroth order approximation is:

$$\begin{aligned}\frac{dX_0}{dt} &= 0,\\ \frac{dY_0}{dt} &= X_0 - (\lambda + X_0)Y_0,\end{aligned}$$

with the ***initial conditions***, $X_0(0) = 1$ and $Y_0(0) = 0$.

Solving first equation gives $X_0(\bar{t}) = c_0 = 1$ from the initial condition.

This leaves the ***linear initial value problem***:

$$\frac{dY_0}{dt} = 1 - (\lambda + 1)Y_0, \qquad Y_0(0) = 0.$$

Inner Approximation 2

Inner Approximation: From the *linear initial value problem*:

$$\frac{dY_0}{d\bar{t}} = 1 - (\lambda+1)Y_0, \qquad Y_0(0) = 0,$$

we obtain the general solution:

$$Y_0(\bar{t}) = c_1 e^{-(\lambda+1)\bar{t}} + \frac{1}{\lambda+1},$$

which with the initial condition gives:

$$Y_0(\bar{t}) = \frac{1}{\lambda+1}\left(1 - e^{-(\lambda+1)\bar{t}}\right).$$

It follows that the *inner approximation* satisfies:

$$\begin{aligned} x_i(t) &= 1, \\ y_i(t) &= \frac{1}{\lambda+1}\left(1 - e^{-(\lambda+1)t/\varepsilon}\right). \end{aligned}$$

Just as with the the *outer approximation*, the *inner approximation* is readily solved. These solutions are combined with our *matching conditions* to obtain a *uniformly converging solution* as $\varepsilon \to 0$.

Matching Condition

Matching Condition: The approximations need to match in the limit as $\varepsilon \to 0$, so for the substrate, x, we have

$$\lim_{t\to 0} x_o(t) = \lim_{\varepsilon \to 0} x_i(t),$$

but $x_i(t) \equiv 1$ and $x_o(t)$ was taken so that $x_o(0) = 1$, which shows that this condition is always satisfied.

Similarly, the approximations need to match in the limit as $\varepsilon \to 0$, so for the complex, y, we have

$$\lim_{t\to 0} y_o(t) = \lim_{\varepsilon \to 0} y_i(t).$$

However, $y_o(t) = \frac{x_o(t)}{\lambda + x_o(t)}$, which clearly converges to $\frac{1}{\lambda+1}$, while for fixed $t > 0$,

$$\lim_{\varepsilon\to 0} y_i(t) = \lim_{\varepsilon\to 0} \frac{1}{\lambda+1}\left(1 - e^{-(\lambda+1)t/\varepsilon}\right) = \frac{1}{\lambda+1}.$$

Uniform Solution

Uniform Solution: The *uniform approximation* is the sum of the *inner* and *outer* approximations minus the common limit:

$$\begin{aligned} x_u(t) &= x_o(t) + 1 - 1 = x_o(t), \\ y_u(t) &= \frac{x_o(t)}{\lambda + x_o(t)} + \frac{1}{\lambda+1}\left(1 - e^{-(\lambda+1)t/\varepsilon}\right) - \frac{1}{\lambda+1}, \\ &= \frac{x_o(t)}{\lambda + x_o(t)} - \frac{1}{\lambda+1} e^{-(\lambda+1)t/\varepsilon}, \end{aligned}$$

where $x_o(t)$ satisfies the implicit equation:

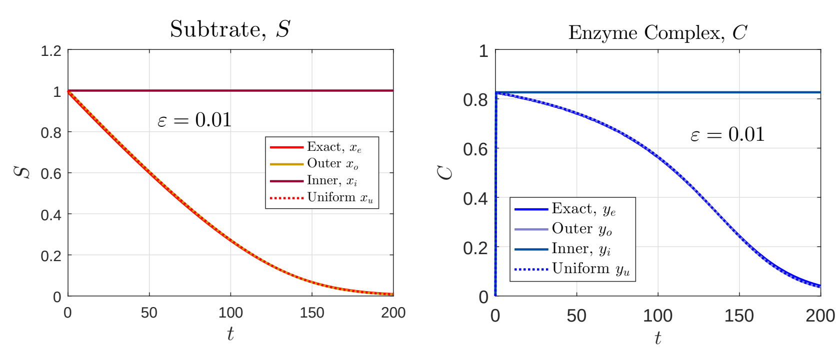
$$x_0(t) + \lambda \ln\left(x_0(t)\right) = (\mu - \lambda)t + 1.$$

Note: That this implicit equation is readily solved for t and is readily solvable for $x_o \in (0, 1)$, which gives an easy method to graph the solution.

However, our **MatLab** program graphing below just integrates the scalar *scalar ODE* for x_o.

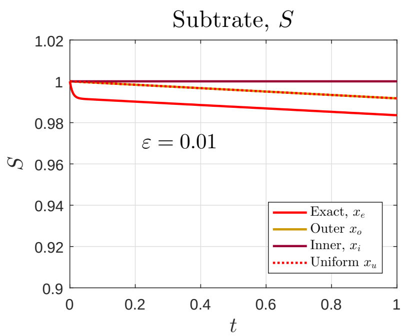
Graphs for Enzyme Problem 1

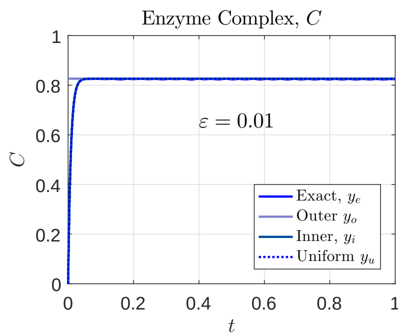
Graphs for Enzyme Problem: Letting $\varepsilon = 0.01$ the graphs below show that the *singular perturbation method* gives very good approximations to the "exact" solution for long term behavior.



Graphs for Enzyme Problem 2

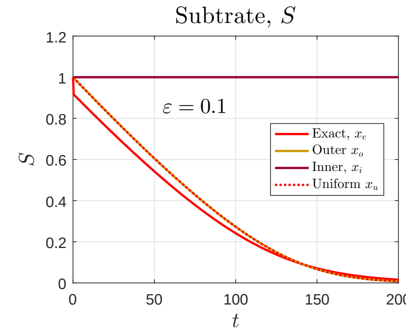
Graphs for Enzyme Problem: Letting $\varepsilon = 0.01$ the graphs also show that the ***singular perturbation method*** gives reasonable approximations to the "exact" solution for early kinetics, failing a bit for the very rapid decline of the substrate initially.

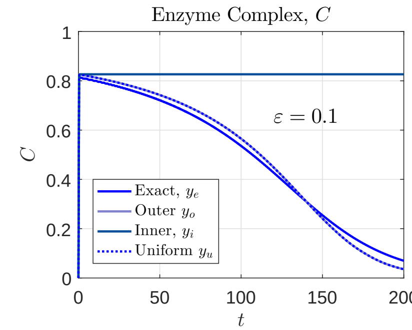




Graphs for Enzyme Problem 3

Graphs for Enzyme Problem: Letting $\varepsilon = 0.1$ the graphs below show that the ***singular perturbation method*** gives good approximations to the "exact" solution for long term behavior, but these approximations separate more with the larger ε.





Graphs for Enzyme Problem 4

Graphs for Enzyme Problem: Letting $\varepsilon = 0.1$ the graphs also show that the ***singular perturbation method*** gives reasonable approximations to the "exact" solution for early kinetics, but failing worse for these approximations separate with the larger ε.