

Math 537 - Ordinary Differential Equations

Lecture Notes – Perturbation Methods

Joseph M. Mahaffy,
(jmahaffy@sdsu.edu)

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

<http://jmahaffy.sdsu.edu>

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Introduction

Mathematical Models rarely have an *explicit form*.

- Require approximation and numerical methods.
- Approximations often use *perturbation methods*:
 - The equations have a *small term*.
 - One physical process is significantly less important than another dominant one.
- Often *rescale* problem with a *small parameter*.
- Use the *small parameter* to create a *Taylor-like series* expansion.
- These methods can be applied to *ODEs*, *PDEs*, *algebraic equations*, and *integral equations*.

Regular Perturbation

Regular Perturbation: For conceptual purposes an *ODE* is used to describe the approach:

$$F(t, y, y', y'', \varepsilon) = 0, \quad t \in I,$$

where t is the independent variable, defined in the interval I , and y is the dependent variable.

The *parameter*, ε , explicitly appears in the equation and is generally considered “small” with

$$\varepsilon \ll 1.$$

It may also occur in the *initial* or *boundary conditions*.

Sometimes a parameter, λ , is “large,” then often introduce

$$\varepsilon = \frac{1}{\lambda} \ll 1.$$

Regular Perturbation

Perturbation Series: Find an ε -*power series* of the solution to the problem with

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots$$

The *regular perturbation method* assumes a solution to the **ODE** in this form, where the functions y_0, y_1, y_2, \dots are found by substituting into the **ODE**.

The first few terms of the ε -*power series* form an approximate solution, called the *perturbation solution* or *approximation*.

Usually only a few terms are necessary.

The method is considered successful if the *approximation* is *uniform*: *i.e.*, the difference between the approximate and exact solutions converges to **zero** at some defined rate as $\varepsilon \rightarrow 0$, uniformly on I (with $\varepsilon < \varepsilon_0$ for some ε_0).

Regular Perturbation

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Regular Perturbation: The dominant behavior comes from the term, $y_0(t)$, the *leading order term*, solving the *unperturbed problem*:

$$F(t, y_0, y_0', y_0'', 0) = 0, \quad t \in I,$$

where this problem is chosen to be solvable.

The term $\varepsilon y_1(t)$, $\varepsilon^2 y_2(t)$, \dots are considered *higher order correction* terms and are assumed to be small relative to the dominant behavior.

Singular Perturbation methods arise when the **regular perturbation** methods fail.

The naive approach often fails for many reasons such as the problem being ill-posed, the solution is invalid on all or parts of the domain, like when there are multiple time or space scales.

Power Series Review

Power Series Review: Some important power series are listed.

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2!}(x - a)^2 + \dots$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

$$(1 + x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots, \quad |x| < 1$$

$$\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots, \quad |x| < 1$$

$$(a + x)^p = a^p + pa^{p-1}x + \frac{p(p-1)}{2!}a^{p-2}x^2 + \frac{p(p-1)(p-2)}{3!}a^{p-3}x^3 + \dots, \quad \left|\frac{x}{a}\right| < 1$$

These power series are commonly used for *asymptotic expansions*.

Quadratic Equation

Quadratic Equation: Consider the equation:

$$x^2 + 2\varepsilon x - 3 = 0, \quad (1)$$

and assume the expansion $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$

The Eqn. (1) satisfies:

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^2 + 2\varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) - 3 = 0,$$

which gives

$$x_0^2 - 3 + \varepsilon(2x_0x_1 + 2x_0) + \varepsilon^2(x_1^2 + 2x_0x_2 + 2x_1) + \mathcal{O}(\varepsilon^3) = 0.$$

Solving each power of ε gives:

$$x_0 = \pm\sqrt{3}, \quad x_1 = -1, \quad x_2 = \pm\frac{1}{2\sqrt{3}}, \quad \dots$$

The approximate solutions are

$$\begin{aligned} x &= \sqrt{3} - \varepsilon + \frac{\varepsilon^2}{2\sqrt{3}} + \dots, \\ x &= -\sqrt{3} - \varepsilon - \frac{\varepsilon^2}{2\sqrt{3}} + \dots \end{aligned}$$

Quadratic Equation

Quadratic Equation; For the equation

$$x^2 + 2\epsilon x - 3 = 0,$$

the quadratic formula gives exact solution;

$$x = -\epsilon \pm \sqrt{3 + \epsilon^2},$$

which can be expanded by the binomial expansion

$$x = -\epsilon \pm \sqrt{3} \pm \frac{\epsilon^2}{2\sqrt{3}} + \dots$$

, agreeing with our asymptotic expansion.

If we take $\epsilon = 0.1$, then the exact and approximate solutions, are:

$$x = -0.1 + \sqrt{3.01} = 1.634935157, \quad x_a = \sqrt{3} - 0.1 + \frac{0.01}{2\sqrt{3}} = 1.634937560,$$

and

$$x = -0.1 - \sqrt{3.01} = -1.834935157, \quad x_a = \sqrt{3} - 0.1 + \frac{0.01}{2\sqrt{3}} = -1.834937560.$$

With $\epsilon = 0.01$, the answers are indistinguishable until the 9th decimal place with

$$x = -0.01 \pm \sqrt{3.0001} = 1.722079675 \quad \text{and} \quad -1.742079675.$$

Transcendental Equation

Transcendental Equation: Consider the equation given by

$$x^3 + \varepsilon \sin(x) + a = 0,$$

which clearly cannot be solved exactly for x .

Take $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ and find the solution to order ε^2 .

Note that

$$\begin{aligned}\sin(x) &= \sin(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) \\ &= \sin(x_0) + \cos(x_0)(\varepsilon x_1 + \varepsilon^2 x_2 + \dots) + \dots \\ &= \sin(x_0) + x_1 \cos(x_0)\varepsilon + \mathcal{O}(\varepsilon^2),\end{aligned}$$

so the equation becomes:

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \mathcal{O}(\varepsilon^3))^3 + \varepsilon(\sin(x_0) + x_1 \cos(x_0)\varepsilon + \mathcal{O}(\varepsilon^2)) + a = 0.$$

Transcendental Equation

Transcendental Equation: From

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \mathcal{O}(\varepsilon^3))^3 + \varepsilon(\sin(x_0) + x_1 \cos(x_0)\varepsilon + \mathcal{O}(\varepsilon^2)) + a = 0,$$

we expand to:

$$x_0^3 + a + (3x_0^2 x_1 + \sin(x_0))\varepsilon + (3x_0^2 x_2 + 3x_0 x_1^2 + x_1 \cos(x_0))\varepsilon^2 + \mathcal{O}(\varepsilon^3) = 0.$$

Solving iteratively gives:

$$x_0 = -a^{\frac{1}{3}}, \quad x_1 = -\frac{\sin(x_0)}{3x_0^2}, \quad x_2 = -\frac{3x_0 x_1^2 + x_1 \cos(x_0)}{3x_0^2}.$$

For $a = 5$, we have $x_0 = -1.709975947$, $x_1 = 0.112896048$, and $x_2 = 0.009239083853$.

With $\varepsilon = 0.1$, then a **3** term expansion of x gives:

$$x = x_0 + 0.1x_1 + 0.01x_2 = -1.698593951.$$

Maple gives a numerical solution to the original equation as $x = -1.698593473$.

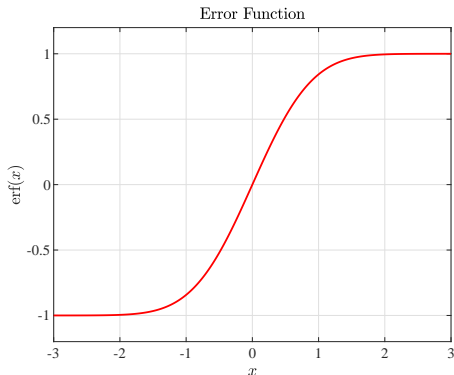
Error Function

1

Error Function: The error function satisfies:

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

In *statistics* with $x > 0$ and a random variable Y that is *normally distributed* with *mean* 0 and *variance* 0.5, the *error function*, $\operatorname{erf}(x)$, describes the *probability* of Y falling in the range $[-x, x]$.



Error Function

Since

$$\lim_{x \rightarrow \infty} \operatorname{erf}(x) = 1,$$

the *complementary error function* is given by:

$$\operatorname{erfc}(x) \equiv 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt.$$

Let $s = t - x$, then

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-(s+x)^2} ds = \frac{2}{\sqrt{\pi}} e^{-x^2} \int_0^{\infty} e^{-(s^2+2sx)} ds.$$

As a first order approximation, we see for $x > 0$

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \int_0^{\infty} e^{-(s^2+2sx)} ds \leq e^{-x^2} \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-s^2} ds = e^{-x^2}.$$

Error Function

Complementary Error Function is better approximated using *integration by parts*:

$$\begin{aligned}\operatorname{erfc}(x) &= \frac{2}{\sqrt{\pi}} e^{-x^2} \int_0^{\infty} e^{-(s^2+2sx)} ds = \frac{2}{\sqrt{\pi}} e^{-x^2} \int_0^{\infty} e^{-s^2} e^{-2sx} ds \\ &= \frac{2}{\sqrt{\pi}} e^{-x^2} \left[-\frac{1}{2x} e^{-s^2} e^{-2sx} \Big|_0^{\infty} + \frac{1}{2x} \int_0^{\infty} (-2s) e^{-s^2} e^{-2sx} ds \right] \\ &= \frac{2}{\sqrt{\pi}} e^{-x^2} \left[\frac{1}{2x} - \frac{1}{x} \int_0^{\infty} s e^{-s^2} e^{-2sx} ds \right] \\ &= \frac{2}{\sqrt{\pi}} e^{-x^2} \left[\frac{1}{2x} - \frac{1}{x} \left(-\frac{s e^{-s^2} e^{-2sx}}{2x} \Big|_0^{\infty} + \frac{1}{2x} \int_0^{\infty} (1-2s^2) e^{-s^2} e^{-2sx} ds \right) \right] \\ &= \frac{2}{\sqrt{\pi}} e^{-x^2} \left[\frac{1}{2x} - \frac{1}{2x^2} \int_0^{\infty} (1-2s^2) e^{-s^2} e^{-2sx} ds \right]\end{aligned}$$

Error Function

Complementary Error Function is approximated for large x by the following:

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x} + \mathcal{O}\left(\frac{e^{-x^2}}{x^2}\right) \quad \text{as } x \rightarrow \infty.$$

	1	2	3	4	5
$\operatorname{erf}(x)$	0.8427	0.9953	0.9999779	0.9999999846	1.0
Approx	0.7924	0.9948	0.9999768	0.9999999841	1.0

A useful *asymptotic expansion* given by Wikipedia is

$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2x^2)^n} \right],$$

which agrees with our computation above.

Motion in Resistive Medium

Resistive ODE: A body mass m with initial velocity V_0 moves in a straight line, where the *resistive force* has magnitude $av - bv^2$ with $v(\tau)$ being the velocity of the object.

It is assumed that $b \ll a$ are constants, then *Newton's Law* gives:

$$m \frac{dv}{d\tau} = -av + bv^2, \quad v(0) = V_0.$$

With the maximum velocity of V_0 and a time scaling based on the decay rate of the linear equation, a/m , we let

$$y = \frac{v}{V_0} \quad \text{and} \quad t = \frac{\tau}{m/a}.$$

This change of variables creates the *dimensionless problem* with $t > 0$:

$$\frac{dy}{dt} = -y + \varepsilon y^2, \quad y(0) = 1,$$

where

$$\varepsilon \equiv \frac{bV_0}{a} \ll 1.$$

This last assumption is that the quadratic resistive force is small compared to the linear force.

Motion in Resistive Medium

Resistive ODE: The scaled model is a Bernoulli's equation and is readily solved exactly.

One makes the substitution $w = y^{-1}$, so $\frac{dw}{dt} = -y^{-2} \frac{dy}{dt}$, so transformed the scaled model becomes:

$$\frac{dw}{dt} - w = -\varepsilon, \quad w(0) = 1.$$

This is a *linear ODE* with the solution:

$$w(t) = e^t \left(1 - \varepsilon \int_0^t e^{-s} ds \right),$$

or

$$w(t) = e^t (1 + \varepsilon(e^{-t} - 1)),$$

or

$$y(t) = \frac{e^{-t}}{1 + \varepsilon(e^{-t} - 1)},$$

which is just a slightly altered form of the *linearized scaled model*.

Motion in Resistive Medium

Perturbation Method: For this solution we assume a solution of the form:

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots,$$

and substitute this into the scaled model giving:

$$y'_0 + \varepsilon y'_1 + \varepsilon^2 y'_2 + \dots = -(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots) + \varepsilon(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots)^2.$$

Collecting powers of ε yields a series of *linear ODEs*:

$$\begin{aligned}y'_0 &= -y_0, \\y'_1 &= -y_1 + y_0^2, \\y'_2 &= -y_2 + 2y_0 y_1, \quad \dots\end{aligned}$$

The *initial condition* satisfies:

$$y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \dots = 1,$$

which gives the sequence of *initial conditions*:

$$y_0(0) = 1, \quad y_1(0) = y_2(0) = \dots = 0.$$

Motion in Resistive Medium

Perturbation Method: The sequence of *linear ODEs* are easily solved to give:

$$\begin{aligned}y_0(t) &= e^{-t}, \\y_1(t) &= e^{-t} - e^{-2t}, \\y_2(t) &= e^{-t} - 2e^{-2t} + e^{-3t}, \dots\end{aligned}$$

It follows that the *approximate solution to $\mathcal{O}(\varepsilon^2)$* satisfies:

$$y_a(t) = e^{-t} + \varepsilon(e^{-t} - e^{-2t}) + \varepsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}).$$

Recall that the exact solution is

$$y(t) = \frac{e^{-t}}{1 + \varepsilon(e^{-t} - 1)},$$

which has a *Taylor series* expansion in ε of

$$y(t) = e^{-t} + \varepsilon(e^{-t} - e^{-2t}) + \varepsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \mathcal{O}(\varepsilon^3),$$

agreeing with the solution from the *perturbation method*.

Note: It is rare that one can obtain the exact solution for comparison.

Kepler's Laws

Two Body Problem: Consider the motion of a planet in the solar system.

- Planet has mass m , and Sun has mass M .
- The position is $\mathbf{r}(t)$.
- *Newton's Law* gives $\mathbf{r}'' = \mathbf{F}$.
- Newton postulated that the *gravitational force* is proportional to the product of the masses and inversely proportional to the square of the distance between the masses.
- In vector form the *Gravitational force* satisfies

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{i}_r,$$

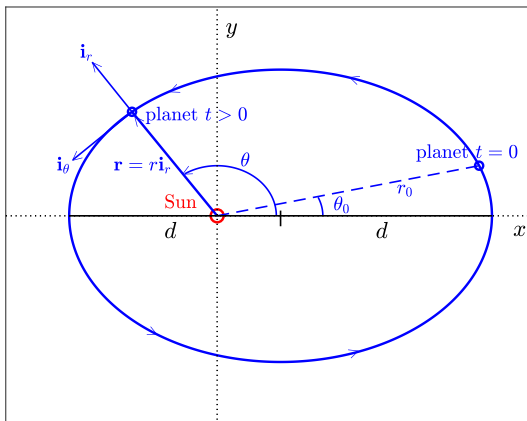
where \mathbf{i}_r is a unit vector in the radial direction and $G = 6.67 \times 10^{-8} \text{cm/g} \cdot \text{s}^2$.

Kepler's Laws

Two Body Problem: From *Newton's Law*, we have the **vector ODE**

$$m\mathbf{r}'' = -\frac{GMm}{r^2}\mathbf{i}_r.$$

Planetary Motion



Kepler's Laws

Two Body Problem: The **vector ODE** satisfies:

$$\mathbf{r}'' = -\frac{GM}{r^2}\mathbf{i}_r,$$

where

$$\mathbf{r}(t) = r(t) \cos(\theta(t))\mathbf{i}_x + r(t) \sin(\theta(t))\mathbf{i}_y = x(t)\mathbf{i}_x + y(t)\mathbf{i}_y.$$

Thus,

$$\begin{aligned}\frac{dx}{dt} &= \cos(\theta) \frac{dr}{dt} - r \sin(\theta) \frac{d\theta}{dt}, \\ \frac{dy}{dt} &= \sin(\theta) \frac{dr}{dt} + r \cos(\theta) \frac{d\theta}{dt}.\end{aligned}$$

It follows from the **vector ODE** that

$$\begin{aligned}\frac{d^2x}{dt^2} &= \cos(\theta) \frac{d^2r}{dt^2} - 2 \sin(\theta) \frac{dr}{dt} \frac{d\theta}{dt} - r \sin(\theta) \frac{d^2\theta}{dt^2} - r \cos(\theta) \left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} \cos(\theta), \\ \frac{d^2y}{dt^2} &= \sin(\theta) \frac{d^2r}{dt^2} + 2 \cos(\theta) \frac{dr}{dt} \frac{d\theta}{dt} + r \cos(\theta) \frac{d^2\theta}{dt^2} - r \sin(\theta) \left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} \sin(\theta).\end{aligned}$$

Kepler's Laws

Two Body Problem: Multiplying the x'' equation by $\cos(\theta)$ and the y'' equation by $\sin(\theta)$ and adding the results gives the *nonlinear ODE*:

$$\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 = -\frac{GM}{r^2}.$$

Similarly, multiplying the x'' equation by $\sin(\theta)$ and the y'' equation by $\cos(\theta)$ and subtracting the results gives the *nonlinear ODE*:

$$r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} = 0.$$

Thus, we have *two nonlinear coupled 2nd order ODEs*, which with the *initial conditions* (4 of them):

$$\mathbf{r}(0) = \mathbf{r}_0 = r_0 \mathbf{i}_{r0} \quad \text{and} \quad \mathbf{r}'(0) = \mathbf{v}_0 = v_{r0} \mathbf{i}_{r0} + v_{\theta 0} \mathbf{i}_{\theta 0},$$

provide a *unique solution* for $r(t)$ and $\theta(t)$, describing the **motion of the planet** for $t > 0$.

Kepler's 2nd Law

1

Kepler's 2nd Law: Take the second *nonlinear ODE*:

$$r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} = 0.$$

and multiply by r , which gives:

$$r^2 \frac{d^2\theta}{dt^2} + 2r \frac{dr}{dt} \frac{d\theta}{dt} = \frac{d(r^2\theta')}{dt} = 0.$$

This is integrated to give

$$r^2 \frac{d\theta}{dt} = \frac{p_0}{m},$$

where p_0 is a constant depending on the initial position and velocity.

The quantity $mr^2\theta'$ is the *angular momentum*, and this result shows that *angular momentum* of the planet is *conserved* (same for all time).

Kepler's 2nd Law

2

Kepler's 2nd Law: Suppose at t_1 the planet is at $P_1 = (r_1, \theta_1)$ and at t_2 the planet is at $P_2 = (r_2, \theta_2)$.

With $\Delta t = t_2 - t_1$, the position vector sweeps a sector with area:

$$\Delta A = \left[\text{Area} \right] = \int_{\theta_1}^{\theta_2} \int_0^{r(\theta)} r \, dr \, d\theta = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta = \frac{1}{2} \int_{t_1}^{t_2} r^2 \theta' \, dt,$$

where $r(\theta)$ is the plane curve traced by the planet's orbit.

From the *conservation of angular momentum* (integrand above being constant), it follows that

$$\Delta A = \frac{p_0}{2m} \Delta t.$$

Property (Kepler's 2nd Law)

The position vector of the planet's orbit sweeps out sectors of equal area in equal time intervals.

The *integral for ΔA* gives more by relating the area with the initial angular momentum per unit mass of the planet.

Kepler's 1st and 3rd Laws

Kepler's 1st and 3rd Laws: From the *conservation of angular momentum* with

$$r^2 \frac{d\theta}{dt} = \frac{p_0}{m} \quad \text{or} \quad \frac{d\theta}{dt} = \frac{p_0}{mr^2},$$

this is substituted into the other *nonlinear ODE* giving:

$$\frac{d^2r}{dt^2} - \frac{p_0^2}{m^2} \frac{1}{r^3} = -\frac{GM}{r^2},$$

which is an *autonomous nonlinear ODE* that can be solved for $r(t)$, but results in a horrendous solution.

The shape of the planet's orbit is obtained by examining $r(\theta)$.

Differentiating and using the *conservation of angular momentum* gives

$$\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} = \frac{p_0}{mr^2} \frac{dr}{d\theta} = -\frac{p_0}{m} \frac{d}{d\theta} \left(\frac{1}{r} \right),$$

so

$$\frac{d^2r}{dt^2} = -\frac{p_0}{m} \frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) \frac{d\theta}{dt} = -\frac{p_0^2}{m^2 r^2} \frac{d^2}{d\theta^2} \left(\frac{1}{r} \right).$$

Kepler's 1st and 3rd Laws

Kepler's 1st Law: It follows that

$$\frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) + \frac{1}{r} = \frac{GMm^2}{p_0^2}.$$

This 2nd order linear ODE has the solution:

$$\frac{1}{r} = A \cos(\theta) + B \sin(\theta) + \frac{GMm^2}{p_0^2}.$$

With the *initial conditions* $r(\theta_0) = r_0$ and $\frac{dr}{d\theta}(\theta_0) = r_0 v_{r0}/v_{\theta 0}$, then for $v_{r0} = 0$ the solution satisfies:

$$r(\theta) = \frac{a(1 - e^2)}{1 + e \cos(\theta - \theta_0)},$$

where

$$\frac{p_0^2}{GMm^2} = a(1 - e^2) \quad \text{and} \quad \frac{p_0^2}{GMm^2 r_0} = 1 + e.$$

For $0 < e < 1$ (*eccentricity* of the ellipse) the expression for $r(\theta)$ gives:

Property (Kepler's 1st Law)

The orbit of the planet around the sun is an ellipse.

Kepler's 1st and 3rd Laws

Kepler's 3rd Law: Let T be the time for the planet to complete one revolution of its orbit, so

$$\frac{p_0 T}{2m} = \text{area of the ellipse} = \pi a^2 \sqrt{1 - e^2} \quad \text{or} \quad T = \frac{2m\pi a^2 \sqrt{1 - e^2}}{p_0}.$$

However,

$$\sqrt{1 - e^2} = \frac{p_0}{m\sqrt{GMa}},$$

so

$$T^2 = \frac{4\pi^2 a^3}{GM}.$$

This gives

Property (Kepler's 3rd Law)

The square of the orbital period is proportional to the third power of the length of the semimajor axis of the elliptical orbit.

Our derivation using **Newton's law** gives a precise prediction of the period for a planet orbiting the sun.

Precession of Perihelion

1

Precession of Perihelion: *Kepler's 1st Law* states that the orbit of a planet is *elliptical*.

- The point of the orbit closest to the *sun* is called *perihelion*, while the furthest point is called *aphelion*.
- Astronomical data show that the *perihelion of Mercury* advances about 574 seconds of arc/century ($\frac{\pi}{2}$ radians = 3.24×10^5 seconds of arc).
- *Newton's law* predicts no advance.
- The multi-body problem using *Newton's law* can account for about 531 seconds of arc/century.
- This leaves about 43 seconds of arc/century, which required *theory of general relativity* because of the speed of the planet *Mercury* and the accumulation of small effects over time.

Precession of Perihelion

Relativistic Effects: Dynamics of planetary motion are complex.

- The *sun* moves, which has a complex effect on the observed motion of the planets.
- This is from complex effects of the relative motion of two moving frames of reference.
- *Relativistic effects* are small except for:
 - When motion is comparable to the speed of light.
 - Time allows the accumulation of many small effects.
- We examine *relativistic effects* by studying the simplified *two-body problem*.
- The initial approximation uses the *Newtonian ODE* derived for studying *Kepler's Laws* to which *relativistic effects* are added.

Precession of Perihelion

Relativistic Effects: Let $r(t)$ be the radial distance between the sun and the planet.

Define the dimensionless variable:

$$\omega_0 = \frac{a(1 - e^2)}{r}, \quad \text{with} \quad a(1 - e^2) = \frac{p_0^2}{GMm^2},$$

then our *Newtonian equation* from before

$$\frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) + \frac{1}{r} = \frac{GMm^2}{p_0^2}.$$

becomes

$$\frac{d^2\omega_0}{d\theta^2} + \omega_0 = 1,$$

which has the solution:

$$\omega_0(\theta) = 1 + e \cos(\theta - \theta_0).$$

Precession of Perihelion

For the theory with the *relativistic effects*, we again define the dimensionless variable, $\omega = \frac{a(1-e^2)}{r}$.

The *relativistic equation* requires a working knowledge of tensor calculus, so we omit derivation of the equations of motion.

It can be shown that the *angular momentum* is still conserved and $\omega(\theta)$ satisfies the equation:

$$\frac{d^2\omega}{d\theta^2} + \omega = 1 + \varepsilon\omega^2, \quad \text{where } \varepsilon = 3 \left(\frac{GMm}{p_0 c} \right)^2,$$

with c being the speed of light.

From before we have $r \approx a(1 - e^2) = \frac{p_0^2}{GMm^2}$, so the *angular velocity* is approximated by:

$$r \frac{d\theta}{dt} = \frac{p_0}{mr} \approx \frac{GMm}{p_0}.$$

From the definition of ε , we find $\sqrt{\varepsilon}$ is a ratio of the planet's speed to the speed of light, which for *Mercury* satisfies:

$$\varepsilon = \mathcal{O}(10^{-9}).$$

Precession of Perihelion

Regular Perturbation: Let

$$\omega(\theta) = \omega_0(\theta) + \varepsilon\omega_1(\theta) + \mathcal{O}(\varepsilon^2),$$

and consider the *ODE in ω* with our perturbation method:

$$\frac{d^2\omega}{d\theta^2} + \omega = 1 + \varepsilon\omega^2.$$

Thus,

$$\omega_0'' + \varepsilon\omega_1'' + \omega_0 + \varepsilon\omega_1 + \mathcal{O}(\varepsilon^2) = 1 + \varepsilon(\omega_0 + \varepsilon\omega_1 + \mathcal{O}(\varepsilon^2))^2.$$

The **zeroth order** terms give:

$$\omega_0'' + \omega_0 = 1,$$

which from the initial value problem before has the solution:

$$\omega_0(\theta) = 1 + e \cos(\theta - \theta_0).$$

Precession of Perihelion

The ε -order terms give the *ODE*:

$$\omega_1'' + \omega_1 = \omega_0^2 = \left(1 + e \cos(\theta - \theta_0)\right)^2 = 1 + \frac{e^2}{2} + \frac{e^2}{2} \cos(2(\theta - \theta_0)) + 2e \cos(\theta - \theta_0).$$

This is solved using the *method of undetermined coefficients* yielding:

$$\omega_1(\theta) = A \cos(\theta) + B \sin(\theta) + 1 + \frac{e^2}{2} - \frac{e^2}{6} \cos(2(\theta - \theta_0)) + e\theta \sin(\theta - \theta_0).$$

The *homogeneous* part and first **3** terms of the *particular solution* are bounded solutions, which remain very small when multiplied by ε .

However the last term comes from *resonance* in this *ODE* and results in an *unbounded solution* (from the θ factor), so results in the dominant behavior over time due to *relativistic effects*.

Precession of Perihelion

Asymptotic Behavior: The *resonance term* is the only significant one in the $\omega_1(\theta)$ solution, so the asymptotic solution is

$$\begin{aligned}\omega(\theta) &\approx 1 + e \left[\cos(\theta - \theta_0) + \varepsilon(\theta - \theta_0) \sin(\theta - \theta_0) \right] \\ &= 1 + e \sqrt{1 + (\varepsilon \bar{\theta})^2} \cos(\bar{\theta} - \phi), \quad \text{with } \bar{\theta} = \theta - \theta_0,\end{aligned}$$

where $\phi = \arctan(\varepsilon \bar{\theta}) \approx \varepsilon \bar{\theta}$ for $|\varepsilon \bar{\theta}| \ll 1$.

The phase angle ϕ varies with θ , (and thus, time).

From this the *perihelion* of the orbit advances by an amount approximately equal to $2\pi\varepsilon = 4.9 \times 10^{-7}$ radian for each revolution of Mercury around the sun, or the *perihelion precesses*.

With an 88-day revolution, Mercury goes through 415 revolutions/century.

Property

*By the relativistic theory, the **perihelion** of Mercury's orbit **precesses** by 43 seconds of arc for each century.*

Spring-Mass Oscillators

Spring-Mass System: Consider a mass, m , connected to a *nonlinear spring* with restoring force $ky + ay^3$ with y being the displacement from equilibrium.

Newton's second law gives:

$$m \frac{d^2 y}{d\tau^2} = -ky - ay^3, \quad \tau > 0,$$

and initial conditions:

$$y(0) = A, \quad \frac{dy}{d\tau}(0) = 0.$$

This problem cannot be solved exactly, but for $a \ll k$, it suggests a *perturbation method*.

From the *initial conditions*, we scale y by the amplitude A .

For the scaling of time, we ignore the cubic term and examine the *ODE*, $my'' + ky = 0$, which has periodic solutions with a frequency of $\sqrt{k/m}$ or period $T = 2\pi\sqrt{m/k}$.

This suggests a scaling of

$$t = \frac{\tau}{\sqrt{m/k}} \quad \text{and} \quad u = \frac{y}{A}.$$

Duffing's Equation

Duffing's Equation: With the previous scaling, the nonlinear mass-spring equation becomes:

$$\ddot{u} + u + \varepsilon u^3 = 0, \quad u(0) = 1, \quad \dot{u}(0) = 0, \quad t > 0,$$

with the “small” dimensionless parameter (assuming $aA^2 \ll k$):

$$\varepsilon \equiv \frac{aA^2}{k} \ll 1.$$

Perturbation method suggests a solution of the form:

$$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \dots$$

This is inserted into the equation above to give:

$$\ddot{u}_0 + \varepsilon \ddot{u}_1 + \varepsilon^2 \ddot{u}_2 + \dots + u_0 + \varepsilon u_1 + \varepsilon^2 u_2(t) + \dots + \varepsilon (u_0 + \varepsilon u_1 + \varepsilon^2 u_2(t) + \dots)^3 = 0.$$

Duffing's Equation

Duffing's Equation: From *IVP* above, we obtain the following sequence of *IVPs*

$$\begin{aligned} \ddot{u}_0 + u_0 &= 0, & u_0(0) &= 1, & \dot{u}_0(0) &= 0, \\ \ddot{u}_1 + u_1 &= -u_0^3, & u_1(0) &= 0, & \dot{u}_1(0) &= 0, \dots \end{aligned}$$

The first *IVP* is easily solved:

$$u_0(t) = \cos(t).$$

To solve the second *IVP*, we use the trig identity $\cos(3t) = 4\cos^3(t) - 3\cos(t)$, so

$$\ddot{u}_1 + u_1 = -\frac{1}{4}(3\cos(t) + \cos(3t)), \quad \text{with } u_1(0) = 0, \quad \dot{u}_1(0) = 0.$$

Once again, the *method of undetermined coefficients* is employed to yield the solution:

$$u_1(t) = \frac{1}{32}(\cos(3t) - \cos(t)) - \frac{3}{8}t \sin(t).$$

The last term (*resonance* again) is clearly unbounded for large t .

This term is called a *secular term* and is inconsistent with the *physical problem*.

Higher order approximations will contain *secular terms* and not cancel out this approximation.

Duffing's Equation

Duffing's Equation is given by:

$$\ddot{u} + u + \varepsilon u^3 = 0, \quad u(0) = 1, \quad \dot{u}(0) = 0, \quad t > 0,$$

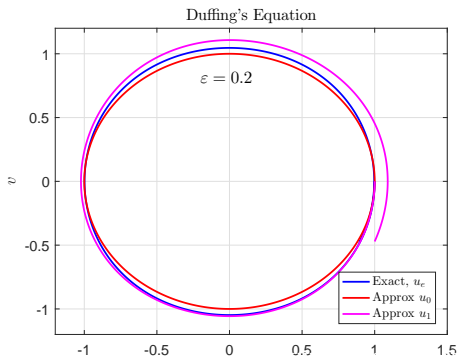
with approximate solutions:

$$u_0(t) = \cos(t).$$

and

$$u_0(t) + \varepsilon u_1(t) = \cos(t) + \varepsilon \left(\frac{1}{32} (\cos(3t) - \cos(t)) - \frac{3}{8} t \sin(t) \right).$$

Below is a *Phase Portrait* of $v(t)$ vs $u(t)$, where $v = \dot{u}$.



Duffing's Equation

```
1 function du = duffDE(t,u)
2 % Duffing's ODE
3 ep = 0.2;
4 du1 = u(2);
5 du2 = -u(1) - ep*(u(1))^3;
6 du = [du1;du2];
7 end
```

```
1 mytitle = 'Duffing's Equation';      % Title
2 xlabel = '$u$';                      % X-label
3 ylabel = '$v$';                      % Y-label
4 tt = linspace(0,2*pi,500);
5 [te,ue] = ode23(@duffDE,tt,[1;0]);
6 ep = 0.2;
7 u0 = cos(tt);
8 v0 = -sin(tt);
```


Duffing's Equation

5

```
9 u1 = cos(tt) + ep*((1/32)*(cos(3*tt)-cos(tt))...
10     -(3/8)*tt.*sin(tt));
11 v1 = -sin(tt) + ep*((1/32)*(-3*sin(3*tt)+sin(tt))...
12     -(3/8)*(tt.*cos(tt)+sin(tt)));
13 plot(ue(:,1),ue(:,2),'b-','LineWidth',1.5);
14 hold on
15 plot(u0,v0,'r-','LineWidth',1.5);
16 plot(u1,v1,'m-','LineWidth',1.5);
17 grid % Adds Gridlines
18 h = legend('Exact, $u_e$', 'Approx $u_0$', ...
19     'Approx $u_1$', 'Location','southeast');
20 set(h,'Interpreter','latex')
21 text(-0.2,0.8,'$\varepsilon = 0.2$', 'FontSize',14,...
22     'interpreter','latex');
23 xlim([-1.2 1.5]);
24 ylim([-1.2 1.2]);
```

Duffing's Equation

```
25 fontlabs = 'Times New Roman';    % Font in labels
26 xlabel(xlab, 'FontSize', 14, 'FontName', fontlabs, ...
27        'interpreter', 'latex');
28 ylabel(ylab, 'FontSize', 14, 'FontName', fontlabs, ...
29        'interpreter', 'latex');
30 title(mytitle, 'FontSize', 16, 'FontName', ...
31        'Times New Roman', 'interpreter', 'latex');
32 set(gca, 'FontSize', 12);          % Axis tick font size
33 print -depsc duff_plot.eps        % Create figure ...
    as EPS file
```

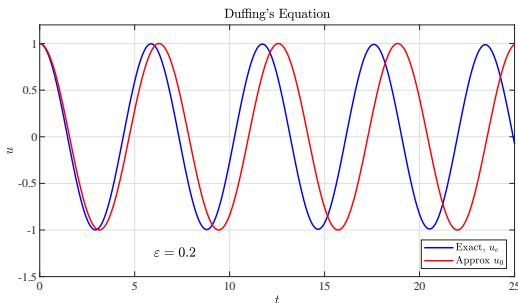
Duffing's Equation Revisited

Duffing's Equation Revisited: The *regular perturbation method* applied to the equation:

$$\ddot{u} + u + \varepsilon u^3 = 0, \quad u(0) = 1, \quad \dot{u}(0) = 0, \quad t > 0,$$

resulted in a *secular term* due to *resonance* in the solution, giving an *unbounded solution*.

The *regular perturbation method* fails to correct for variations in the period, which accumulate over time leading to solutions *out-of-phase*.



Poincaré-Lindstedt Method

Poincaré-Lindstedt Method: With *Duffing's equation* a perturbation is introduced into the time scale.

Specifically, let

$$u(\tau) = u_0(\tau) + \varepsilon u_1(\tau) + \varepsilon^2 u_2(\tau) + \dots,$$

where

$$\tau = \omega t \quad \text{with} \quad \omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \dots,$$

so $\omega_0 = 1$, matching the frequency of the unperturbed problem.

This scaling changes *Duffing's equation* (with ' differentiation with respect to τ) to

$$\omega^2 u'' + u + \varepsilon u^3 = 0, \quad u(0) = 1, \quad u'(0) = 0, \quad \tau > 0,$$

Substitution gives:

$$(1 + \varepsilon \omega_1 + \dots)^2 (u_0'' + \varepsilon u_1'' + \dots) + (u_0 + \varepsilon u_1 + \dots) + \varepsilon (u_0 + \varepsilon u_1 + \dots)^3 = 0,$$

and

$$u_0(0) + \varepsilon u_1(0) + \dots = 1, \quad u_0'(0) + \varepsilon u_1'(0) + \dots = 0.$$

Poincaré-Lindstedt Method

Poincaré-Lindstedt Method: Collect coefficients of the powers of ε to give the following differential equations:

$$u_0'' + u_0 = 0, \quad u_0(0) = 1, \quad u_0'(0) = 0,$$

and

$$u_1'' + u_1 = -2\omega_1 u_0'' - u_0^3, \quad u_1(0) = u_1'(0) = 0, \dots$$

The solution to the *first equation* is:

$$u_0(\tau) = \cos(\tau),$$

so the *second DE equation* becomes:

$$u_1'' + u_1 = 2\omega_1 \cos(\tau) - \cos^3(\tau) = \left(2\omega_1 - \frac{3}{4}\right) \cos(\tau) - \frac{1}{4} \cos(3\tau).$$

Since $\cos(\tau)$ is a solution to the *homogeneous equation*, this term on the right side leads to a particular solution, leading to a *secular term* of the form $\tau \cos(\tau)$.

This term is eliminated by taking $\omega_1 = \frac{3}{8}$.

Poincaré-Lindstedt Method

Poincaré-Lindstedt Method: With $\omega_1 = \frac{3}{8}$, the *second DE equation* becomes:

$$u_1'' + u_1 = -\frac{1}{4} \cos(3\tau), \quad u_1(0) = u_1'(0) = 0.$$

The general solution is given by:

$$u_1(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau) + \frac{1}{32} \cos(3\tau),$$

which with the initial conditions gives:

$$u_1(\tau) = \frac{1}{32} (\cos(3\tau) - \cos(\tau)).$$

Thus, a *first-order, uniformly valid perturbation solution* is

$$u(\tau) = \cos(\tau) + \frac{\varepsilon}{32} (\cos(3\tau) - \cos(\tau)) + \dots,$$

where

$$\tau = t + \frac{3\varepsilon}{8}t + \dots$$

Poincaré-Lindstedt Method

Duffing's Equation is given by:

$$\ddot{u} + u + \varepsilon u^3 = 0, \quad u(0) = 1, \quad \dot{u}(0) = 0, \quad t > 0,$$

with the Poincaré-Lindstedt approximate solution:

$$u(\tau) = \cos(\tau) + \frac{\varepsilon}{32} (\cos(3\tau) - \cos(\tau)) + \dots,$$

where

$$\tau = t + \frac{3\varepsilon}{8}t + \dots$$

Below is a *Phase Portrait* and *Time Series* of $v(t)$ vs $u(t)$, where $v = \dot{u}$.

