# Math 537 - Ordinary Differential Equations

## Lecture Notes – Perturbation Methods

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## Outline







## Introduction

**Mathematical Models** rarely have an *explicit form*.

- Require approximation and numerical methods.
- Approximations often use ***perturbation methods***:
  - The equations have a *small term*.
  - One physical process is significantly less important than another dominant one.
- Often *rescale* problem with a *small parameter*.
- Use the *small parameter* to create a *Taylor-like series* expansion.
- These methods can be applied to ***ODEs***, ***PDEs***, ***algebraic equations***, and ***integral equations***.





## Regular Perturbation 1

**Regular Perturbation**: For conceptual purposes an ***ODE*** is used to describe the approach:

$$F(t, y, y', y'', \varepsilon) = 0, \qquad t \in I,$$

where $t$ is the independent variable, defined in the interval $I$, and $y$ is the dependent variable.

The *parameter*, $\varepsilon$, explicitly appears in the equation and is generally considered "small" with

$$\varepsilon \ll 1.$$

It may also occur in the ***initial*** or ***boundary conditions***.

Sometimes a parameter, $\lambda$, is "large," then often introduce

$$\varepsilon = \frac{1}{\lambda} \ll 1.$$

## Regular Perturbation 2

**Perturbation Series**: Find an *$\varepsilon$-power series* of the solution to the problem with

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots$$

The *regular perturbation method* assumes a solution to the *ODE* in this form, where the functions $y_0$, $y_1$, $y_2$, ... are found by substituting into the *ODE*.

The first few terms of the *$\varepsilon$-power series* form an approximate solution, called the *perturbation solution* or *approximation*.

Usually only a few terms are necessary.

The method is considered successful if the *approximation* is *uniform*: *i.e.*, the difference between the approximate and exact solutions converges to **zero** at some defined rate as $\varepsilon \to 0$, uniformly on $I$ (with $\varepsilon < \varepsilon_0$ for some $\varepsilon_0$).





## Regular Perturbation 3

**Regular Perturbation**: The dominant behavior comes from the term, $y_0(t)$, the *leading order term*, solving the *unperturbed problem*:

$$F(t, y_0, y_0', y_0'', 0) = 0, \qquad t \in I,$$

where this problem is chosen to be solvable.

The term $\varepsilon y_1(t)$, $\varepsilon^2 y_2(t)$, ... are considered *higher order correction* terms and are assumed to be small relative to the dominant behavior.

*Singular Perturbation* methods arise when the **regular perturbation** methods fail.

The naive approach often fails for many reasons such as the problem being ill-posed, the solution is invalid on all or parts of the domain, like when there are multiple time or space scales.





## Power Series Review

**Power Series Review**: Some important power series are listed.

$$\begin{aligned}
f(x) &= f(a) + f'(a)(x-a) + \tfrac{1}{2!}(x-a)^2 + \dots \\
e^x &= 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \dots \\
\sin(x) &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \dots \\
\cos(x) &= 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \dots \\
(1+x)^p &= 1 + px + \tfrac{p(p-1)}{2!}x^2 + \tfrac{p(p-1)(p-2)}{3!}x^3 + \dots, \qquad |x| < 1 \\
\ln(1+x) &= x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots, \qquad |x| < 1 \\
(a+x)^p &= a^p + pa^{p-1}x + \tfrac{p(p-1)}{2!}a^{p-2}x^2 + \tfrac{p(p-1)(p-2)}{3!}a^{p-3}x^3 + \dots, \qquad \left|\tfrac{x}{a}\right| < 1
\end{aligned}$$

These power series are commonly used for *asymptotic expansions*.





## Quadratic Equation 1

**Quadratic Equation**: Consider the equation:

$$x^2 + 2\varepsilon x - 3 = 0, \tag{1}$$

and assume the expansion $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$

The Eqn. (1) satisfies:

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^2 + 2\varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) - 3 = 0,$$

which gives

$$x_0^2 - 3 + \varepsilon(2x_0x_1 + 2x_0) + \varepsilon^2(x_1^2 + 2x_0x_2 + 2x_1) + \mathcal{O}(\varepsilon^3) = 0.$$

Solving each power of $\varepsilon$ gives:

$$x_0 = \pm\sqrt{3}, \qquad x_1 = -1, \qquad x_2 = \pm\frac{1}{2\sqrt{3}}, \qquad \dots$$

The approximate solutions are

$$\begin{aligned}
x &= \sqrt{3} - \varepsilon + \frac{\varepsilon^2}{2\sqrt{3}} + \dots, \\
x &= -\sqrt{3} - \varepsilon - \frac{\varepsilon^2}{2\sqrt{3}} + \dots.
\end{aligned}$$

## Quadratic Equation 2

**Quadratic Equation**; For the equation

$$x^2 + 2\varepsilon x - 3 = 0,$$

the quadratic formula gives exact solution;

$$x = -\varepsilon \pm \sqrt{3 + \varepsilon^2},$$

which can be expanded by the binomial expansion

$$x = -\varepsilon \pm \sqrt{3} \pm \frac{\varepsilon^2}{2\sqrt{3}} + \dots .$$

, agreeing with our asymptotic expansion.

If we take $\varepsilon = 0.1$, then the exact and approximate solutions, are:

$$x = -0.1 + \sqrt{3.01} = 1.634935157, \qquad x_a = \sqrt{3} - 0.1 + \frac{0.01}{2\sqrt{3}} = 1.634937560,$$

and

$$x = -0.1 - \sqrt{3.01} = -1.834935157, \qquad x_a = \sqrt{3} - 0.1 + \frac{0.01}{2\sqrt{3}} = -1.834937560.$$

With $\varepsilon = 0.01$, the answers are indistinguishable until the $9^{th}$ decimal place with

$$x = -0.01 \pm \sqrt{3.0001} = 1.722079675 \quad \text{and} \quad -1.742079675.$$





## Transcendental Equation 1

**Transcendental Equation**: Consider the equation given by

$$x^3 + \varepsilon \sin(x) + a = 0,$$

which clearly cannot be solved exactly for $x$.

Take $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ and find the solution to order $\varepsilon^2$.

Note that

$$\begin{aligned}\sin(x) &= \sin(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) \\ &= \sin(x_0) + \cos(x_0)\big(\varepsilon x_1 + \varepsilon^2 x_2 + \dots\big) + \dots \\ &= \sin(x_0) + x_1 \cos(x_0)\varepsilon + \mathcal{O}\left(\varepsilon^2\right),\end{aligned}$$

so the equation becomes:

$$\big(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \mathcal{O}\left(\varepsilon^3\right)\big)^3 + \varepsilon\big(\sin(x_0) + x_1\cos(x_0)\varepsilon + \mathcal{O}\left(\varepsilon^2\right)\big) + a = 0.$$





## Transcendental Equation 2

**Transcendental Equation**: From

$$\big(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \mathcal{O}\left(\varepsilon^3\right)\big)^3 + \varepsilon\big(\sin(x_0) + x_1\cos(x_0)\varepsilon + \mathcal{O}\left(\varepsilon^2\right)\big) + a = 0,$$

we expand to:

$$x_0^3 + a + \left(3x_0^2 x_1 + \sin(x_0)\right)\varepsilon + \left(3x_0^2 x_2 + 3x_0 x_1^2 + x_1\cos(x_0)\right)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) = 0.$$

Solving iteratively gives:

$$x_0 = -a^{\frac{1}{3}}, \qquad x_1 = -\frac{\sin(x_0)}{3x_0^2}, \qquad x_2 = -\frac{3x_0 x_1^2 + x_1\cos(x_0)}{3x_0^2}.$$

For $a = 5$, we have $x_0 = -1.709975947$, $x_1 = 0.112896048$, and $x_2 = 0.009239083853$.

With $\varepsilon = 0.1$, then a **3** term expansion of $x$ gives:

$$x = x_0 + 0.1x_1 + 0.01x_2 = -1.698593951.$$

Maple gives a numerical solution to the original equation as $x = -1.698593473$.



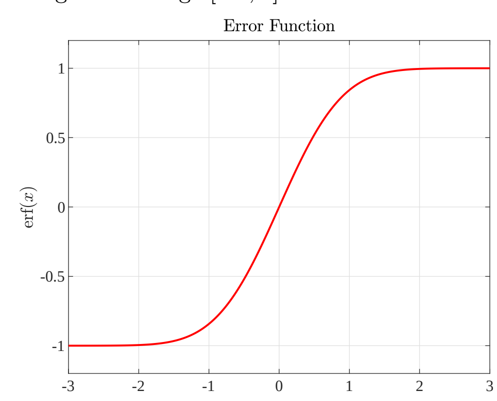


## Error Function 1

**Error Function**: The error function satisfies:

$$\text{erf}(x) = \frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

In ***statistics*** with $x > 0$ and a random variable $Y$ that is ***normally distributed*** with ***mean*** 0 and ***variance*** 0.5, the ***error function***, $\text{erf}(x)$, describes the ***probability*** of $Y$ falling in the range $[-x, x]$.



## Error Function 2

Since

$$\lim_{x\to\infty} \operatorname{erf}(x) = 1,$$

the ***complementary error function*** is given by:

$$\operatorname{erfc}(x) \equiv 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$

Let $s = t - x$, then

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-(s+x)^2} ds = \frac{2}{\sqrt{\pi}} e^{-x^2} \int_0^\infty e^{-(s^2+2sx)} ds.$$

As a first order approximation, we see for $x > 0$

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \int_0^\infty e^{-(s^2+2sx)} ds \le e^{-x^2} \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} ds = e^{-x^2}.$$





## Error Function 3

**Complementary Error Function** is better approximated using ***integration by parts***:

$$\begin{aligned}
\operatorname{erfc}(x) &= \tfrac{2}{\sqrt{\pi}} e^{-x^2} \int_0^\infty e^{-(s^2+2sx)} ds = \tfrac{2}{\sqrt{\pi}} e^{-x^2} \int_0^\infty e^{-s^2} e^{-2sx} ds \\
&= \tfrac{2}{\sqrt{\pi}} e^{-x^2} \left[ -\frac{1}{2x} e^{-s^2} e^{-2sx} \Big|_0^\infty + \frac{1}{2x} \int_0^\infty (-2s) e^{-s^2} e^{-2sx} ds \right] \\
&= \tfrac{2}{\sqrt{\pi}} e^{-x^2} \left[ \frac{1}{2x} - \frac{1}{x} \int_0^\infty s e^{-s^2} e^{-2sx} ds \right] \\
&= \tfrac{2}{\sqrt{\pi}} e^{-x^2} \left[ \frac{1}{2x} - \frac{1}{x} \left( -\frac{s e^{-s^2} e^{-2sx}}{2x} \Big|_0^\infty + \frac{1}{2x} \int_0^\infty (1-2s^2) e^{-s^2} e^{-2sx} ds \right) \right] \\
&= \tfrac{2}{\sqrt{\pi}} e^{-x^2} \left[ \frac{1}{2x} - \frac{1}{2x^2} \int_0^\infty (1-2s^2) e^{-s^2} e^{-2sx} ds \right]
\end{aligned}$$





## Error Function 4

**Complementary Error Function** is approximated for large $x$ by the following:

$$\operatorname{erfc}(x) = \tfrac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x} + \mathcal{O}\left( \frac{e^{-x^2}}{x^2} \right) \qquad \text{as} \quad x \to \infty.$$

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\operatorname{erf}(x)$ | 0.8427 | 0.9953 | 0.9999779 | 0.9999999846 | 1.0 |
| Approx | 0.7924 | 0.9948 | 0.9999768 | 0.9999999841 | 1.0 |

A useful ***asymptotic expansion*** given by Wikipedia is

$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \left[ 1 + \sum_{n=1}^\infty (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2x^2)^n} \right],$$

which agrees with our computation above.





## Motion in Resistive Medium 1

**Resistive ODE**: A body mass $m$ with initial velocity $V_0$ moves in a straight line, where the ***resistive force*** has magnitude $av - bv^2$ with $v(\tau)$ being the velocity of the object.

It is assumed that $b \ll a$ are constants, then ***Newton's Law*** gives:

$$m\tfrac{dv}{d\tau} = -av + bv^2, \qquad v(0) = V_0.$$

With the maximum velocity of $V_0$ and a time scaling based on the decay rate of the linear equation, $a/m$, we let

$$y = \tfrac{v}{V_0} \qquad \text{and} \qquad t = \tfrac{\tau}{m/a}.$$

This change of variables creates the ***dimensionless problem*** with $t > 0$:

$$\tfrac{dy}{dt} = -y + \varepsilon y^2, \qquad y(0) = 1,$$

where

$$\varepsilon \equiv \tfrac{bV_0}{a} \ll 1.$$

This last assumption is that the quadratic resistive force is small compared to the linear force.

## Motion in Resistive Medium



**Resistive ODE**: The scaled model is a Bernoulli's equation and is readily solved exactly.

One makes the substitution $w = y^{-1}$, so $\frac{dw}{dt} = -y^{-2}\frac{dy}{dt}$, so transformed the scaled model becomes:

$$\frac{dw}{dt} - w = -\varepsilon, \qquad w(0) = 1.$$

This is a ***linear ODE*** with the solution:

$$w(t) = e^t\left(1 - \varepsilon\int_0^t e^{-s}ds\right),$$

or

$$w(t) = e^t\left(1 + \varepsilon(e^{-t} - 1)\right),$$

or

$$y(t) = \frac{e^{-t}}{1 + \varepsilon(e^{-t} - 1)},$$

which is just a slightly altered form of the ***linearized scaled model***.





## Motion in Resistive Medium



**Perturbation Method**: For this solution we assume a solution of the form:

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots,$$

and substitute this into the scaled model giving:

$$y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots = -(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots) + \varepsilon(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots)^2.$$

Collecting powers of $\varepsilon$ yields a series of ***linear ODEs***:

$$\begin{aligned} y_0' &= -y_0, \\ y_1' &= -y_1 + y_0^2, \\ y_2' &= -y_2 + 2y_0y_1, \quad \dots \end{aligned}$$

The ***initial condition*** satisfies:

$$y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \cdots = 1,$$

which gives the sequence of ***initial conditions***:

$$y_0(0) = 1, \qquad y_1(0) = y_2(0) = \cdots = 0.$$





## Motion in Resistive Medium



**Perturbation Method**: The sequence of ***linear ODEs*** are easily solved to give:

$$\begin{aligned} y_0(t) &= e^{-t}, \\ y_1(t) &= e^{-t} - e^{-2t}, \\ y_2(t) &= e^{-t} - 2e^{-2t} + e^{-3t}, \quad \dots \end{aligned}$$

It follows that the *approximate solution to $\mathcal{O}(\varepsilon^2)$* satisfies:

$$y_a(t) = e^{-t} + \varepsilon\left(e^{-t} - e^{-2t}\right) + \varepsilon^2\left(e^{-t} - 2e^{-2t} + e^{-3t}\right).$$

Recall that the exact solution is

$$y(t) = \frac{e^{-t}}{1 + \varepsilon(e^{-t} - 1)},$$

which has a *Taylor series* expansion in $\varepsilon$ of

$$y(t) = e^{-t} + \varepsilon\left(e^{-t} - e^{-2t}\right) + \varepsilon^2\left(e^{-t} - 2e^{-2t} + e^{-3t}\right) + \mathcal{O}(\varepsilon^3),$$

agreeing with the solution from the *perturbation method*.

**Note**: It is rare that one can obtain the exact solution for comparison.





## Kepler's Laws



**Two Body Problem**: Consider the motion of a planet in the solar system.

- Planet has mass $m$, and Sun has mass $M$.
- The position is $\mathbf{r}(t)$.
- ***Newton's Law*** gives $\mathbf{r}'' = \mathbf{F}$.
- Newton postulated that the ***gravitational force*** is proportional to the product of the masses and inversely proportional to the square of the distance between the masses.
- In vector form the ***Gravitational force*** satisfies

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{i}_r,$$

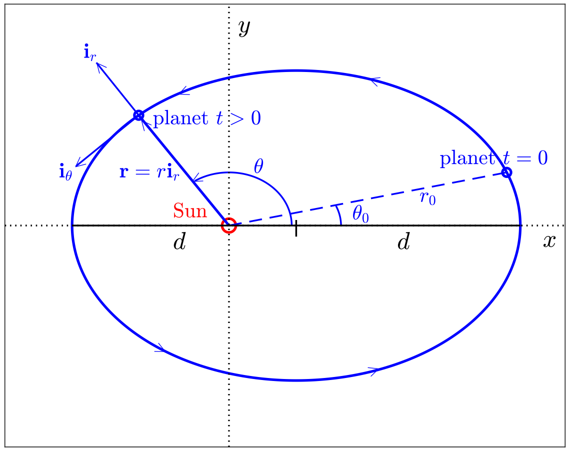
where $\mathbf{i}_r$ is a unit vector in the radial direction and $G = 6.67 \times 10^{-8}\text{cm/g}\cdot\text{s}^2$.

## Kepler's Laws 2

**Two Body Problem**: From ***Newton's Law***, we have the **vector ODE**

$$m\mathbf{r}'' = -\frac{GMm}{r^2}\mathbf{i}_r.$$

Planetary Motion








## Kepler's Laws 3

**Two Body Problem**: The **vector ODE** satisfies:

$$\mathbf{r}'' = -\frac{GM}{r^2}\mathbf{i}_r,$$

where

$$\mathbf{r}(t) = r(t)\cos(\theta(t))\mathbf{i}_x + r(t)\sin(\theta(t))\mathbf{i}_y = x(t)\mathbf{i}_x + y(t)\mathbf{i}_y.$$

Thus,

$$\begin{aligned}\frac{dx}{dt} &= \cos(\theta)\frac{dr}{dt} - r\sin(\theta)\frac{d\theta}{dt},\\ \frac{dy}{dt} &= \sin(\theta)\frac{dr}{dt} + r\cos(\theta)\frac{d\theta}{dt}.\end{aligned}$$

It follows from the **vector ODE** that

$$\begin{aligned}\frac{d^2x}{dt^2} &= \cos(\theta)\frac{d^2r}{dt^2} - 2\sin(\theta)\frac{dr}{dt}\frac{d\theta}{dt} - r\sin(\theta)\frac{d^2\theta}{dt^2} - r\cos(\theta)\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2}\cos(\theta),\\ \frac{d^2y}{dt^2} &= \sin(\theta)\frac{d^2r}{dt^2} + 2\cos(\theta)\frac{dr}{dt}\frac{d\theta}{dt} + r\cos(\theta)\frac{d^2\theta}{dt^2} - r\sin(\theta)\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2}\sin(\theta).\end{aligned}$$





## Kepler's Laws 4

**Two Body Problem**: Multiplying the $x''$ equation by $\cos(\theta)$ and the $y''$ equation by $\sin(\theta)$ and adding the results gives the ***nonlinear ODE***:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2}.$$

Similarly, multiplying the $x''$ equation by $\sin(\theta)$ and the $y''$ equation by $\cos(\theta)$ and subtracting the results gives the ***nonlinear ODE***:

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0.$$

Thus, we have ***two nonlinear coupled $2^{nd}$ order ODEs***, which with the ***initial conditions*** (**4** of them):

$$\mathbf{r}(0) = \mathbf{r}_0 = r_0\mathbf{i}_{r0} \quad \text{and} \quad \mathbf{r}'(0) = \mathbf{v}_0 = v_{r0}\mathbf{i}_{r0} + v_{\theta 0}\mathbf{i}_{\theta 0},$$

provide a ***unique solution*** for $r(t)$ and $\theta(t)$, describing the **motion of the planet** for $t > 0$.





## Kepler's $2^{nd}$ Law 1

**Kepler's $2^{nd}$ Law**: Take the second ***nonlinear ODE***:

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0.$$

and multiply by $r$, which gives:

$$r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt} = \frac{d(r^2\theta')}{dt} = 0.$$

This is integrated to give

$$r^2\frac{d\theta}{dt} = \frac{p_0}{m},$$

where $p_0$ is a constant depending on the initial position and velocity.

The quantity $mr^2\theta'$ is the ***angular momentum***, and this result shows that ***angular momentum*** of the planet is ***conserved*** (same for all time).

## Kepler's $2^{nd}$ Law 2

**Kepler's $2^{nd}$ Law**: Suppose at $t_1$ the planet is at $P_1 = (r_1, \theta_1)$ and at $t_2$ the planet is at $P_2 = (r_2, \theta_2)$.

With $\Delta t = t_2 - t_1$, the position vector sweeps a sector with area:

$$\Delta A = \left[\text{Area}\right] = \int_{\theta_1}^{\theta_2} \int_0^{r(\theta)} r\, dr\, d\theta = \tfrac{1}{2} \int_{\theta_1}^{\theta_2} r^2 d\theta = \tfrac{1}{2} \int_{t_1}^{t_2} r^2 \theta' dt,$$

where $r(\theta)$ is the plane curve traced by the planet's orbit.

From the *conservation of angular momentum* (integrand above being constant), it follows that

$$\Delta A = \frac{p_0}{2m} \Delta t.$$

**Property (Kepler's $2^{nd}$ Law)**

*The position vector of the planet's orbit sweeps out sectors of equal area in equal time intervals.*

The *integral for $\Delta A$* gives more by relating the area with the initial angular momentum per unit mass of the planet.





## Kepler's $1^{st}$ and $3^{rd}$ Laws 1

**Kepler's $1^{st}$ and $3^{rd}$ Laws**: From the *conservation of angular momentum* with

$$r^2 \frac{d\theta}{dt} = \frac{p_0}{m} \qquad \text{or} \qquad \frac{d\theta}{dt} = \frac{p_0}{mr^2},$$

this is substituted into the other *nonlinear ODE* giving:

$$\frac{d^2r}{dt^2} - \frac{p_0^2}{m^2}\frac{1}{r^3} = -\frac{GM}{r^2},$$

which is an *autonomous nonlinear ODE* that can be solved for $r(t)$, but results in a horrendous solution.

The shape of the planet's orbit is obtained by examining $r(\theta)$.

Differentiating and using the *conservation of angular momentum* gives

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{p_0}{mr^2}\frac{dr}{d\theta} = -\frac{p_0}{m}\frac{d}{d\theta}\left(\frac{1}{r}\right),$$

so

$$\frac{d^2r}{dt^2} = -\frac{p_0}{m}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\frac{d\theta}{dt} = -\frac{p_0^2}{m^2r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right).$$





## Kepler's $1^{st}$ and $3^{rd}$ Laws 3

**Kepler's $1^{st}$ Law**: It follows that

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{GMm^2}{p_0^2}.$$

This $2^{nd}$ *order linear ODE* has the solution:

$$\frac{1}{r} = A\cos(\theta) + B\sin(\theta) + \frac{GMm^2}{p_0^2}.$$

With the *initial conditions* $r(\theta_0) = r_0$ and $\frac{dr}{d\theta}(\theta_0) = r_0 v_{r0}/v_{\theta 0}$, then for $v_{r0} = 0$ the solution satisfies:

$$r(\theta) = \frac{a(1-e^2)}{1 + e\cos(\theta - \theta_0)},$$

where

$$\frac{p_0^2}{GMm^2} = a(1-e^2) \qquad \text{and} \qquad \frac{p_0^2}{GMm^2 r_0} = 1 + e.$$

For $0 < e < 1$ (*eccentricity* of the ellipse) the expression for $r(\theta)$ gives:

**Property (Kepler's $1^{st}$ Law)**

*The orbit of the planet around the sun is an ellipse.*





## Kepler's $1^{st}$ and $3^{rd}$ Laws 3

**Kepler's $3^{rd}$ Law**: Let $T$ be the time for the planet to complete one revolution of its orbit, so

$$\frac{p_0 T}{2m} = \text{area of the ellipse} = \pi a^2\sqrt{1-e^2} \quad \text{or} \quad T = \frac{2m\pi a^2\sqrt{1-e^2}}{p_0}.$$

However,

$$\sqrt{1-e^2} = \frac{p_0}{m\sqrt{GMa}},$$

so

$$T^2 = \frac{4\pi^2 a^3}{GM}.$$

This gives

**Property (Kepler's $3^{rd}$ Law)**

*The square of the orbital period is proportional to the third power of the length of the semimajor axis of the elliptical orbit.*

Our derivation using *Newton's law* gives a precise prediction of the period for a planet orbiting the sun.

## Precession of Perihelion 1

**Precession of Perihelion**: ***Kepler's*** $1^{st}$ ***Law*** states that the orbit of a planet is ***elliptical***.

- The point of the orbit closest to the ***sun*** is called ***perihelion***, while the furthest point is called ***aphelion***.
- Astronomical data show that the ***perihelion of Mercury*** advances about 574 seconds of arc/century ($\frac{\pi}{2}$ radians $= 3.24 \times 10^5$ seconds of arc).
- ***Newton's law*** predicts no advance.
- The multi-body problem using ***Newton's law*** can account for about 531 seconds of arc/century.
- This leaves about 43 seconds of arc/century, which required ***theory of general relativity*** because of the speed of the planet ***Mercury*** and the accumulation of small effects over time.





## Precession of Perihelion 2

**Relativistic Effects**: Dynamics of planetary motion are complex.

- The ***sun*** moves, which has a complex effect on the observed motion of the planets.
- This is from complex effects of the relative motion of two moving frames of reference.
- ***Relativistic effects*** are small except for:
  - When motion is comparable to the speed of light.
  - Time allows the accumulation of many small effects.
- We examine ***relativistic effects*** by studying the simplified ***two-body problem***.
- The initial approximation uses the ***Newtonian ODE*** derived for studying ***Kepler's Laws*** to which ***relativistic effects*** are added.





## Precession of Perihelion 3

**Relativistic Effects**: Let $r(t)$ be the radial distance between the sun and the planet.

Define the dimensionless variable:

$$\omega_0 = \frac{a(1-e^2)}{r}, \qquad \text{with} \quad a(1-e^2) = \frac{p_0^2}{GMm^2},$$

then our ***Newtonian equation*** from before

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{GMm^2}{p_0^2}.$$

becomes

$$\frac{d^2\omega_0}{d\theta^2} + \omega_0 = 1,$$

which has the solution:

$$\omega_0(\theta) = 1 + e\cos(\theta - \theta_0).$$





## Precession of Perihelion 4

For the theory with the ***relativistic effects***, we again define the dimensionless variable, $\omega = \frac{a(1-e^2)}{r}$.

The ***relativistic equation*** requires a working knowledge of tensor calculus, so we omit derivation of the equations of motion.

It can be shown that the ***angular momentum*** is still conserved and $\omega(\theta)$ satisfies the equation:

$$\frac{d^2\omega}{d\theta^2} + \omega = 1 + \varepsilon\omega^2, \qquad \text{where} \quad \varepsilon = 3\left(\frac{GMm}{p_0 c}\right)^2,$$

with $c$ being the speed of light.

From before we have $r \approx a(1-e^2) = \frac{p_0^2}{GMm^2}$, so the ***angular velocity*** is approximated by:

$$r\frac{d\theta}{dt} = \frac{p_0}{mr} \approx \frac{GMm}{p_0}.$$

From the definition of $\varepsilon$, we find $\sqrt{\varepsilon}$ is a ratio of the planet's speed to the speed of light, which for ***Mercury*** satisfies:

$$\varepsilon = \mathcal{O}\left(10^{-9}\right).$$

## Precession of Perihelion



**Regular Perturbation**: Let

$$\omega(\theta) = \omega_0(\theta) + \varepsilon\omega_1(\theta) + \mathcal{O}\left(\varepsilon^2\right),$$

and consider the ***ODE in $\omega$*** with our perturbation method:

$$\frac{d^2\omega}{d\theta^2} + \omega = 1 + \varepsilon\omega^2.$$

Thus,

$$\omega_0'' + \varepsilon\omega_1'' + \omega_0 + \varepsilon\omega_1 + \mathcal{O}\left(\varepsilon^2\right) = 1 + \varepsilon\left(\omega_0 + \varepsilon\omega_1 + \mathcal{O}\left(\varepsilon^2\right)\right)^2.$$

The **zeroth order** terms give:

$$\omega_0'' + \omega_0 = 1,$$

which from the initial value problem before has the solution:

$$\omega_0(\theta) = 1 + e\cos(\theta - \theta_0).$$





## Precession of Perihelion



The $\varepsilon$-order terms give the ***ODE***:

$$\omega_1'' + \omega_1 = \omega_0^2 = \Big(1 + e\cos(\theta - \theta_0)\Big)^2 = 1 + \frac{e^2}{2} + \frac{e^2}{2}\cos(2(\theta - \theta_0)) + 2e\cos(\theta - \theta_0).$$

This is solved using the ***method of undetermined coefficients*** yielding:

$$\omega_1(\theta) = A\cos(\theta) + B\sin(\theta) + 1 + \frac{e^2}{2} - \frac{e^2}{6}\cos(2(\theta - \theta_0)) + e\theta\sin(\theta - \theta_0).$$

The ***homogeneous*** part and first **3** terms of the ***particular solution*** are bounded solutions, which remain very small when multiplied by $\varepsilon$.

However the last term comes from ***resonance*** in this ***ODE*** and results in an ***unbounded solution*** (from the $\theta$ factor), so results in the dominant behavior over time due to ***relativistic effects***.





## Precession of Perihelion



**Asymptotic Behavior**: The ***resonance term*** is the only significant one in the $\omega_1(\theta)$ solution, so the asymptotic solution is

$$\begin{aligned}\omega(\theta) &\approx 1 + e\Big[\cos(\theta - \theta_0) + \varepsilon(\theta - \theta_0)\sin(\theta - \theta_0)\Big] \\ &= 1 + e\sqrt{1 + (\varepsilon\bar{\theta})^2}\cos(\bar{\theta} - \phi), \qquad \text{with} \quad \bar{\theta} = \theta - \theta_0,\end{aligned}$$

where $\phi = \arctan(\varepsilon\bar{\theta}) \approx \varepsilon\bar{\theta}$ for $|\varepsilon\bar{\theta}| \ll 1$.

The phase angle $\phi$ varies with $\theta$, (and thus, time).

From this the ***perihelion*** of the orbit advances by an amount approximately equal to $2\pi\varepsilon = 4.9 \times 10^{-7}$ radian for each revolution of Mercury around the sun, or the ***perihelion precesses***.

With an 88-day revolution, Mercury goes through 415 revolutions/century.

**Property**

*By the relativistic theory, the **perihelion** of Mercury's orbit **precesses** by 43 seconds of arc for each century.*





## Spring-Mass Oscillators

**Spring-Mass System**: Consider a mass, $m$, connected to a ***nonlinear spring*** with restoring force $ky + ay^3$ with $y$ being the displacement from equilibrium.

***Newton's second law*** gives:

$$m\frac{d^2y}{d\tau^2} = -ky - ay^3, \qquad \tau > 0,$$

and initial conditions:

$$y(0) = A, \qquad \frac{dy}{d\tau}(0) = 0.$$

This problem cannot be solved exactly, but for $a \ll k$,it suggests a ***perturbation method***.

From the ***initial conditions***, we scale $y$ by the amplitude $A$.

For the scaling of time, we ignore the cubic term and examine the ***ODE***, $my'' + ky = 0$, which has periodic solutions with a frequency of $\sqrt{k/m}$ or period $T = 2\pi\sqrt{m/k}$.

This suggests a scaling of

$$t = \frac{\tau}{\sqrt{m/k}} \qquad \text{and} \qquad u = \frac{y}{A}.$$

# Duffing's Equation



**Duffing's Equation**: With the previous scaling, the nonlinear mass-spring equation becomes:

$$\ddot{u} + u + \varepsilon u^3 = 0, \qquad u(0) = 1, \quad \dot{u}(0) = 0, \qquad t > 0,$$

with the "small" dimensionless parameter (assuming $aA^2 \ll k$):

$$\varepsilon \equiv \frac{aA^2}{k} \ll 1.$$

***Perturbation method*** suggests a solution of the form:

$$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \dots$$

This is inserted into the equation above to give:

$$\ddot{u}_0 + \varepsilon\ddot{u}_1 + \varepsilon^2\ddot{u}_2 + \dots + u_0 + \varepsilon u_1 + \varepsilon^2 u_2(t) + \dots + \varepsilon(u_0 + \varepsilon u_1 + \varepsilon^2 u_2(t) + \dots)^3 = 0.$$





# Duffing's Equation



**Duffing's Equation**: From ***IVP*** above, we obtain the following sequence of ***IVPs***

$$\ddot{u}_0 + u_0 = 0, \qquad u_0(0) = 1, \qquad \dot{u}_0(0) = 0,$$
$$\ddot{u}_1 + u_1 = -u_0^3, \qquad u_1(0) = 0, \qquad \dot{u}_1(0) = 0, \dots$$

The first ***IVP*** is easily solved:

$$u_0(t) = \cos(t).$$

To solve the second ***IVP***, we use the trig identity $\cos(3t) = 4\cos^3(t) - 3\cos(t)$, so

$$\ddot{u}_1 + u_1 = -\tfrac{1}{4}\big(3\cos(t) + \cos(3t)\big), \qquad \text{with} \quad u_1(0) = 0, \quad \dot{u}_1(0) = 0.$$

Once again, the ***method of undetermined coefficients*** is employed to yield the solution:

$$u_1(t) = \tfrac{1}{32}\big(\cos(3t) - \cos(t)\big) - \tfrac{3}{8}t\,\sin(t).$$

The last term (***resonance*** again) is clearly unbounded for large $t$.

This term is called a ***secular term*** and is inconsistent with the ***physical problem***.

Higher order approximations will contain ***secular terms*** and not cancel out this approximation.





# Duffing's Equation



**Duffing's Equation** is given by:

$$\ddot{u} + u + \varepsilon u^3 = 0, \qquad u(0) = 1, \quad \dot{u}(0) = 0, \qquad t > 0,$$

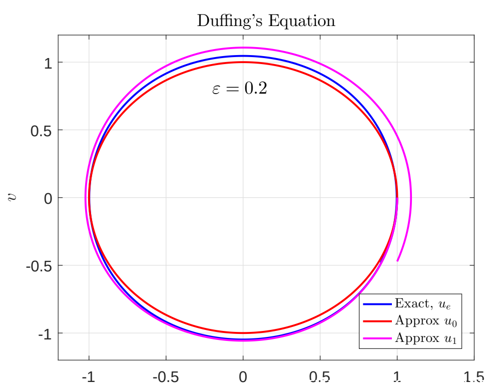
with approximate solutions:

$$u_0(t) = \cos(t).$$

and

$$u_0(t) + \varepsilon u_1(t) = \cos(t) + \varepsilon\left(\tfrac{1}{32}\big(\cos(3t) - \cos(t)\big) - \tfrac{3}{8}t\,\sin(t)\right).$$

Below is a ***Phase Portrait*** of $v(t)$ vs $u(t)$, where $v = \dot{u}$.








# Duffing's Equation



```
function du = duffDE(t,u)
% Duffing's ODE
ep = 0.2;
du1 = u(2);
du2 = -u(1) - ep*(u(1))^3;
du = [du1;du2];
end
```

```
mytitle = 'Duffing''s Equation';     % Title
xlab = '$u$';                        % X-label
ylab = '$v$';                        % Y-label
tt = linspace(0,2*pi,500);
[te,ue] = ode23(@duffDE,tt,[1;0]);
ep = 0.2;
u0 = cos(tt);
v0 = -sin(tt);
```

## Duffing's Equation



```
u1 = cos(tt) + ep*((1/32)*(cos(3*tt)-cos(tt))...
    -(3/8)*tt.*sin(tt));
v1 = -sin(tt) + ep*((1/32)*(-3*sin(3*tt)+sin(tt))...
     -(3/8)*(tt.*cos(tt)+sin(tt)));
plot(ue(:,1),ue(:,2),'b-','LineWidth',1.5);
hold on
plot(u0,v0,'r-','LineWidth',1.5);
plot(u1,v1,'m-','LineWidth',1.5);
grid                                % Adds Gridlines
h =legend('Exact, $u_e$', 'Approx $u_0$',...
    'Approx $u_1$', 'Location','southeast');
set(h,'Interpreter','latex')
text(-0.2,0.8,'$\varepsilon = 0.2$','FontSize',14,...
    'interpreter','latex');
xlim([-1.2 1.5]);
ylim([-1.2 1.2]);
```





## Duffing's Equation



```
fontlabs = 'Times New Roman';    % Font in labels
xlabel(xlab,'FontSize',14,'FontName',fontlabs,...
    'interpreter','latex');
ylabel(ylab,'FontSize',14,'FontName',fontlabs,...
    'interpreter','latex');
title(mytitle,'FontSize',16,'FontName',...
    'Times New Roman','interpreter','latex');
set(gca,'FontSize',12);          % Axis tick font size
print -depsc duff_plot.eps       % Create figure ...
    as EPS file
```





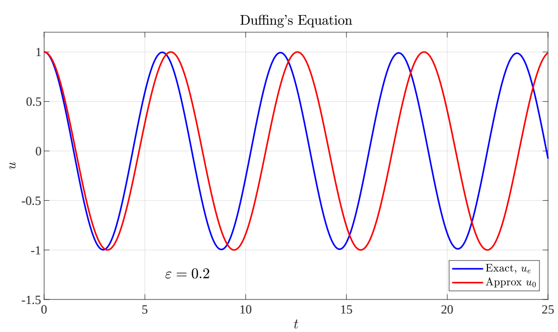
## Duffing's Equation Revisited

**Duffing's Equation Revisited**: The *regular perturbation method* applied to the equation:

$$\ddot{u} + u + \varepsilon u^3 = 0, \qquad u(0) = 1, \quad \dot{u}(0) = 0, \qquad t > 0,$$

resulted in a *secular term* due to ***resonance*** in the solution, giving an ***unbounded solution***.

The *regular perturbation method* fails to correct for variations in the period, which accumulate over time leading to solutions ***out-of-phase***.








## Poincaré-Lindstedt Method



**Poincaré-Lindstedt Method**: With *Duffing's equation* a perturbation is introduced into the time scale.

Specifically, let

$$u(\tau) = u_0(\tau) + \varepsilon u_1(\tau) + \varepsilon^2 u_2(\tau) + \dots,$$

where

$$\tau = \omega t \qquad \text{with} \qquad \omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots,$$

so $\omega_0 = 1$, matching the frequency of the unperturbed problem.

This scaling changes *Duffing's equation* (with $'$ differentiation with respect to $\tau$) to

$$\omega^2 u'' + u + \varepsilon u^3 = 0, \qquad u(0) = 1, \quad u'(0) = 0, \qquad \tau > 0,$$

Substitution gives:

$$(1 + \varepsilon\omega_1 + \dots)^2(u_0'' + \varepsilon u_1'' + \dots) + (u_0 + \varepsilon u_1 + \dots) + \varepsilon(u_0 + \varepsilon u_1 + \dots)^3 = 0,$$

and

$$u_0(0) + \varepsilon u_1(0) + \dots = 1, \qquad u_0'(0) + \varepsilon u_1'(0) + \dots = 0.$$

## Poincaré-Lindstedt Method



**Poincaré-Lindstedt Method**: Collect coefficients of the powers of $\varepsilon$ to give the following differential equations:

$$u_0'' + u_0 = 0, \qquad u_0(0) = 1, \qquad u_0'(0) = 0,$$

and

$$u_1'' + u_1 = -2\omega_1 u_0'' - u_0^3, \qquad u_1(0) = u_1'(0) = 0, \dots$$

The solution to the ***first equation*** is:

$$u_0(\tau) = \cos(\tau),$$

so the ***second DE equation*** becomes:

$$u_1'' + u_1 = 2\omega_1\cos(\tau) - \cos^3(\tau) = \left(2\omega_1 - \tfrac{3}{4}\right)\cos(\tau) - \tfrac{1}{4}\cos(3\tau).$$

Since $\cos(\tau)$ is a solution to the ***homogeneous equation***, this term on the right side leads to a particular solution, leading to a ***secular term*** of the form $\tau\cos(\tau)$.

This term is eliminated by taking $\omega_1 = \frac{3}{8}$.





## Poincaré-Lindstedt Method



**Poincaré-Lindstedt Method**: With $\omega_1 = \frac{3}{8}$, the ***second DE equation*** becomes:

$$u_1'' + u_1 = -\tfrac{1}{4}\cos(3\tau), \qquad u_1(0) = u_1'(0) = 0.$$

The general solution is given by:

$$u_1(\tau) = c_1\cos(\tau) + c_2\sin(\tau) + \tfrac{1}{32}\cos(3\tau),$$

which with the initial conditions gives:

$$u_1(\tau) = \tfrac{1}{32}\big(\cos(3\tau) - \cos(\tau)\big).$$

Thus, a ***first-order, uniformly valid perturbation solution*** is

$$u(\tau) = \cos(\tau) + \tfrac{\varepsilon}{32}\big(\cos(3\tau) - \cos(\tau)\big) + \dots,$$

where

$$\tau = t + \tfrac{3\varepsilon}{8}t + \dots$$





## Poincaré-Lindstedt Method



**Duffing's Equation** is given by:

$$\ddot{u} + u + \varepsilon u^3 = 0, \qquad u(0) = 1, \quad \dot{u}(0) = 0, \qquad t > 0,$$

with the Poincaré-Lindstedt approximate solution:

$$u(\tau) = \cos(\tau) + \tfrac{\varepsilon}{32}\big(\cos(3\tau) - \cos(\tau)\big) + \dots,$$

where

$$\tau = t + \tfrac{3\varepsilon}{8}t + \dots$$

Below is a ***Phase Portrait*** and ***Time Series*** of $v(t)$ vs $u(t)$, where $v = \dot{u}$.