Math 537 - Ordinary Differential Equations

Lecture Notes – Method of Averaging

Joseph M. Mahaffy,
⟨jmahaffy@sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://jmahaffy.sdsu.edu

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Outline

Introduction

Method of Averaging is a useful tool in ***dynamical systems***, where *time-scales* in a ***differential equation*** are separated between a ***fast oscillation*** and ***slower behavior***.

- The fast oscillations are *averaged out* to allow the determination of the ***qualitative behavior*** of averaged dynamical system.
- The averaging method dates from perturbation problems that arose in ***celestial mechanics***.
- This method dates back to 1788, when Lagrange formulated the ***gravitational three-body problem*** as a perturbation of the ***two-body problem***.
- The validity of this method waited until Fatou (1928) proved some of the asymptotic results.
- Significant results, including Krylov-Bogoliubov, followed in the 1930s, making ***averaging methods*** important **classical tools for analyzing nonlinear oscillations**.

Introduction

The **Method of Averaging** is applicable to systems of the form:

$$\dot{x} = \varepsilon f(x, t, \varepsilon), \qquad x \in U \subseteq \mathbb{R}^n, \quad \varepsilon \ll 1,$$

where $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^n$ is C^r, $r \geq 1$ bounded on bounded sets, and of period $T > 0$ in t, and U is bounded and open. The associated autonomous averaged system is defined as

$$\dot{y} = \frac{\varepsilon}{T}\int_0^T f(y, t, 0)dt \equiv \varepsilon \bar{f}(y).$$

The ***averaging method*** approximates the original system in x by the averaged system in y, which is presumably easier to study.

- ***Qualitative analysis*** giving the dynamics of the averaged system provides information about the properties of the dynamics for the original system.
- The solution y provides approximate values for x over finite time that is inversely proportional to the slow time scale, $1/\varepsilon$.
- The asymptotic behavior of the original system is captured by the dynamical equation for y
- This allows the ***qualitative methods for autonomous dynamical systems*** to analyze the equilibria and more complex structures, such as slow manifold and invariant manifolds, as well as their stability in the phase space of the averaged system.

Seasonal Logistic Growth

Example - Seasonal Logistic Growth: Consider the *logistic growth model* with some seasonal variation:

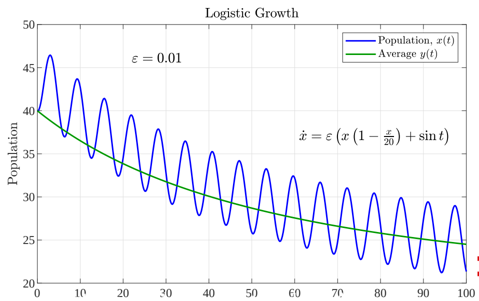
$$\dot{x} = \varepsilon\left(x\left(1 - \frac{x}{M}\right) + \sin(\omega t)\right), \qquad x \in \mathbb{R}, \quad 0 < \varepsilon \ll 1.$$

It follows that the averaged equation satisfies:

$$\dot{y} = \varepsilon y\left(1 - \frac{y}{M}\right), \qquad y \in \mathbb{R}.$$

The solution $x(t)$ shows *complicated dynamics*.

However, when the oscillations are removed, the solution $y(t)$ reduces to a simple case of a ***stable equilibrium*** at $y_e = M$ and an ***unstable equilibrium*** at $y_e = 0$.



Background - Linear Theory 1

Linear Systems: Earlier we studied the linear system:

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n,$$

and showed we could make a transformation $x = Py$, so that $P^{-1}AP = J$ was in *Jordan canonical form*.

Specifically, this decoupled the system in y based on the ***eigenvalues*** of A, and we observed the different behaviors from the ***fundamental solution set***, $y(t) = e^{Jt}$, which transformed back to the ***fundamental solution set*** of the original system:

$$\Phi(t) = e^{At}, \qquad \text{which gave unique solutions} \qquad \phi_t(x_0) = x(x_0, t) = e^{At}x_0.$$

This ***fundamental solution*** generates a ***flow***: $e^{At}x_0 : \mathbb{R}^n \to \mathbb{R}^n$, which gives all the solutions to $\dot{x} = Ax$.

Specifically, the ***linear subspaces*** spanned by the ***eigenvectors*** of A are ***invariant*** under the ***flow***, $\phi_t(x_0) = e^{At}x_0$.

The ***Jordan canonical form*** helps visualize the distinct behaviors of the ***ODE***, $\dot{y} = Jy$ in a "nice" ***orthogonal set***.

Background - Linear Theory 2

Linear Systems: For the linear system:

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n,$$

the matrix has n eigenvalues, which allowed finding n (generalized) eigenvectors.

The ***eigenspaces*** of A are ***invariant subspaces*** for the flow, $\phi_t(x_0) = e^{At}x_0$.

Motivated by the ***Jordan canonical form***, we divide the subspaces spanned by the eigenvectors into **three classes:**

1. The ***stable subspace***, $E^s = \text{span}\{v^1, \dots, v^{n_s}\}$,
2. The ***unstable subspace***, $E^u = \text{span}\{u^1, \dots, u^{n_u}\}$,
3. The ***center subspace***, $E^c = \text{span}\{w^1, \dots, w^{n_c}\}$,

where $v^1, \dots, v^{n_s}$ are the n_s (generalized) eigenvectors whose eigenvalues have **negative real parts**, $u^1, \dots, u^{n_u}$ are the n_u (generalized) eigenvectors whose eigenvalues have **positve real parts**, and $w^1, \dots, w^{n_c}$ are the n_c (generalized) eigenvectors whose eigenvalues have **zero real parts**.

Clearly, $n_s + n_u + n_c = n$, and the names reflect the behavior of the ***flows*** on the particular subspaces with those on E^s exponentially decaying, E^u exponentially growing, and E^c doing neither.

Background - Nonlinear Theory 1

Nonlinear Systems: We extend these stability ideas from the linear system to the nonlinear autonomous problem

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \qquad x(0) = x_0. \tag{1}$$

The ***nonlinear system*** has ***existence-uniqueness*** is some small neighborhood of $t = 0$ near x_0 provided adequate smoothness of f.

Equilibria: As always, one starts with the ***fixed points*** or ***equilibria*** of (1) by solving $f(x_e) = 0$, which may be **nontrivial**.

Linearization: Assume that x_e is a ***fixed point*** of (1), then to characterize the behavior of solutions to (1), we examine the ***linearization*** at x_e and create the linear system:

$$\dot{\xi} = Df(x_e)\xi, \qquad \xi \in \mathbb{R}^n,$$

where $Df = [\partial f_i/\partial x_j]$ is the ***Jacobian matrix*** of the first partial derivatives of $f = [f_1(x_1, \dots, x_n), f_2(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)]^T$ and $x = x_e + \xi$ with $\xi \ll 1$.

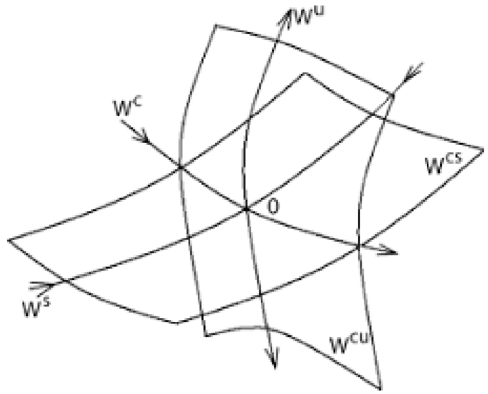
The ***linearized flow map*** near x_e is given by:

$$D\phi_t(x_e)\xi = e^{tDf(x_e)}\xi.$$

Background - Nonlinear Theory

Ideally, we would like to decompose our space of flows at least locally (near a ***fixed point***) into the behaviors similar to the ones observed for the ***linear system***, which was decomposed into the ***stable subspace***, E^s, the ***unstable subspace***, E^u, and the ***center subspace***, E^c.

We expect the ***nonlinearity*** to curve our subspaces, but below gives the decomposition of the ***flows*** desired.



Background - Important Theorems

Theorem (Hartman-Grobman)

*If $Df(x_e)$ has no zero or purely imaginary eigenvalues, then there is a **homeomorphism**, h, defined on some neighborhood, U, of $x_e \in \mathbb{R}^n$ locally taking orbits of the **nonlinear flow**, ϕ_t of (1) to those of the **linear flow**, $e^{tDf(x_e)}\xi$. The **homeomorphism** preserves the sense of the orbits and can be chosen to preserve parametrization by time.*

Definition (Hyperbolic Fixed Point)

When $Df(x_e)$ has no eigenvalues with **zero real part**, x_e is called a ***hyperbolic*** or ***nondegenerate fixed point***.

The behavior of solutions of (1) near a ***hyperbolic fixed point*** is determined (locally) by the linearization.

Background - Example

Example: Consider the ***ODE*** given by:

$$\ddot{x} + \varepsilon x^2 \dot{x} + x = 0,$$

which is easily rewritten as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \varepsilon \begin{pmatrix} 0 \\ x_1^2 x_2 \end{pmatrix}.$$

This system has an ***equilibrium***, $(x_{1e}, x_{2e}) = (0, 0)$.

The ***linearized system*** has eigenvalues, $\lambda = \pm i$, which have **zero real part**.

This results in a ***center*** for $\varepsilon = 0$.

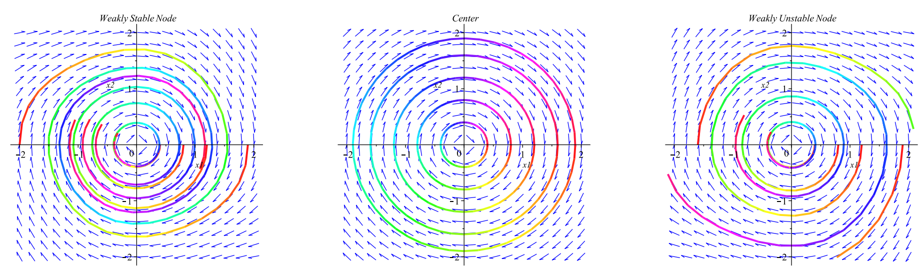
However, if $\varepsilon > 0$, then the system results in a ***nonhyperbolic*** or ***weak attracting sink***.

If $\varepsilon < 0$, then the system results in a ***nonhyperbolic*** or ***weak attracting source***.

Background - Example

Example: Phase plots for the ***ODE***

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \varepsilon \begin{pmatrix} 0 \\ x_1^2 x_2 \end{pmatrix}.$$



$\varepsilon = 0.2$ $\qquad$ $\varepsilon = 0$ $\qquad$ $\varepsilon = -0.2$

Background - Manifolds

Manifolds: For *linear systems* we obtained *invariant subspaces* spanning $\mathbb{R}^n$ for stable, unstable, and center behavior.

For the *nonlinear ODE* the behavior can only be defined locally, so we define the *local stable and unstable manifolds*.

Definition (Local Stable and Unstable Manifold)

Define the *local stable and unstable manifolds* of the *fixed point*, x_e, $W^s_{loc}(x_e)$, $W^u_{loc}(x_e)$, as follows:

- $W^s_{loc}(x_e) = \{x \in U | \phi_t(x) \to x_e \text{ as } t \to \infty, \text{ and } \phi_t(x) \in U \text{ for all } t \geq 0\}$,
- $W^u_{loc}(x_e) = \{x \in U | \phi_t(x) \to x_e \text{ as } t \to -\infty, \text{ and } \phi_t(x) \in U \text{ for all } t \leq 0\}$,

where $U \subset \mathbb{R}^n$ is a neighborhood of the *fixed point*, x_e.

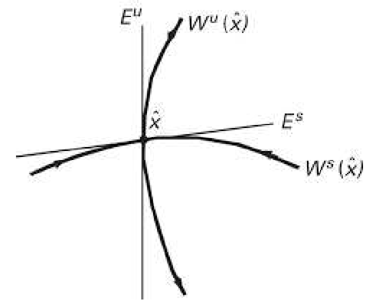
These *invariant manifolds*, $W^s_{loc}(x_e)$ and $W^u_{loc}(x_e)$, provide nonlinear analogues of the flat stable and unstable eigenspaces, E^s and E^u of the linear problem.

Stable Manifold Theorem

The *Stable Manifold Theorem* shows that $W^s_{loc}(x_e)$ and $W^u_{loc}(x_e)$ are tangent to the eigenspaces, E^s and E^u.

Theorem (Stable Manifold Theorem)

Suppose that $\dot{x} = f(x)$ has a ***hyperbolic fixed point****, x_e. Then there exist* ***local stable and unstable manifolds****, $W^s_{loc}(x_e)$ and $W^u_{loc}(x_e)$, of the same dimensions, n_s and n_u, as those of the* ***eigenspaces****, E^s and E^u, of the linearized system and tangent to E^s and E^u at x_e. $W^s_{loc}(x_e)$ and $W^u_{loc}(x_e)$ are as smooth as the function, f.*



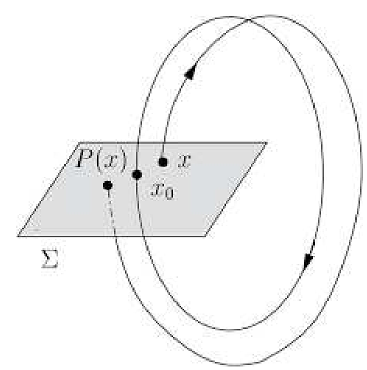
Stable Manifold Theorem

Stable Manifold Theorem: Below we make a number of comments about the *nonlinear ODE* with respect to this theorem.

- This theorem avoids discussion about a *center manifold* being tangent to E^c, confining the results to *hyperbolic fixed points*.
- Interest in a *center manifold* often relates to studies in *bifurcation theory*.
- The *local invariant manifolds* have global analogues.
  - The *global stable manifold*, W^s, follows points in $W^s_{loc}(x_e)$ flow backwards in time:
  $$W^s(x_e) = \bigcup_{t \leq 0} \phi_t(W^s_{loc}(x_e)).$$
  - The *global unstable manifold*, W^u, follows points in $W^u_{loc}(x_e)$ flow forward in time:
  $$W^u(x_e) = \bigcup_{t \geq 0} \phi_t(W^u_{loc}(x_e)).$$
- Existence and uniqueness ensures that two stable (unstable) manifolds of distinct fixed points, x_{1e}, x_{2e}, cannot intersect.
- Intersections of stable and unstable manifolds of distinct fixed points or the same fixed point can occur.
- These intersections are often the source of complex dynamics, such as chaos.

Poincaré Maps 1

Poincaré Maps: A *first recurrence map* or *Poincaré map* is the intersection of a periodic orbit for the *flow*, ϕ_t, of an *ODE* in $\mathbb{R}^n$ with a particular lower-dimensional subspace, called the *Poincaré section*, transversal to the flow of the system.



Poincaré Maps

Definition (Poincaré Map)

Let γ be a periodic orbit of some flow $\phi_t(x_0) \in \mathbb{R}^n$ arising from some ***ODE***. Let $\Sigma \subset \mathbb{R}^n$ be a local differentiable section of dimension $n-1$, where the flow ϕ_t is everywhere ***transverse*** to Σ, called a ***Poincaré section*** through x_0 (implying that if n_v is the normal to Σ at a point x, then $n_v \cdot \phi_t \neq 0$).

Given an open and connected neighborhood $U \subset \Sigma$ of x_0, a function

$$P : U \to \Sigma$$

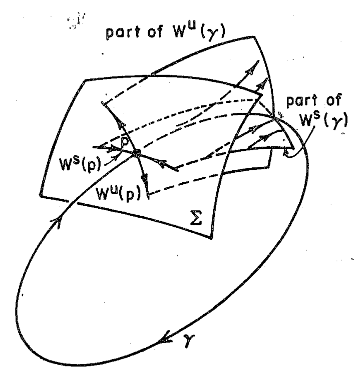
is called a ***Poincaré map*** for the orbit γ on the ***Poincaré section*** Σ through the point x_0 if:

- $P(x_0) = x_0$.
- $P(U)$ is a neighborhood of x_0 and $P : U \to P(U)$ is a ***diffeomorphism***.
- For every point x in U, the ***positive semi-orbit*** of x intersects Σ for the first time at $P(x)$

Poincaré Maps

Poincaré maps can be interpreted as ***discrete dynamical systems*** (Math 538).

The ***stability of a periodic orbit*** of the original ***ODE*** connects to the ***stability of the fixed point*** of the corresponding ***Poincaré map***.



Poincaré Maps

Poincaré maps have the property that the periodic orbit γ of the continuous dynamical system, ***ODE***, is ***stable*** if and only if the ***fixed point*** x_0 of the discrete dynamical system is ***stable***.

Let the ***Poincaré map***, $P : U \to \Sigma$, be defined as above and create a ***discrete dynamical system***,

$$P(n, x) \equiv P^n(x) \qquad \text{with} \qquad P : \mathbb{Z}^n \times U \to U,$$

where

$$P^0 \equiv \mathrm{id}_U, \qquad P^{n+1} \equiv P \circ P^n, \qquad P^{-n-1} \equiv P^{-1} \circ P^{-n}$$

and x_0 is a ***fixed point***.

Stability of this discrete map is found by ***linearizing***, P, at x_0, and determining the ***eigenvalues*** of $DP(x_0)$.

If these ***eigenvalues*** are all inside the unit circle, then x_0 is ***stable***, which in turn gives the periodic orbit of the ***ODE*** as being ***stable***.

Nonautonomous ODE

Nonautonomous ODE: Consider the ODE system:

$$\dot{x} = f(x, t), \qquad (x, t) \in \mathbb{R}^n \times \mathbb{R},$$

where $f(\cdot, t) = f(\cdot, t + T)$ is T-periodic.

This is written as an ***autonomous ODE*** by making time an explicit state variable:

$$\begin{aligned} \dot{x} &= f(x, \theta), \\ \dot{\theta} &= 1, \qquad (x, \theta) \in \mathbb{R}^n \times S^1. \end{aligned}$$

The phase space is the manifold $\mathbb{R}^n \times S^1$, where the circular component $S^1 = \mathbb{R}(\text{mod } T)$ reflects the periodicity of the vector field in θ of this ODE.

In this case we obtain a natural ***global cross section***

$$\Sigma = \{(x, \theta) \in \mathbb{R}^n \times S^1 | \theta = \theta_0\},$$

and the ***Poincaré map*** $P : \Sigma \to \Sigma$ is defined globally by

$$P(x_0) = \Pi[\phi_T(x_0, \theta_0)],$$

where $\phi_t : \mathbb{R}^n \times S^1 \to \mathbb{R}^n \times S^1$ is the ***flow*** of the ODE and Π denotes the projection onto the $x \in \mathbb{R}^n$ phase space at $\theta = \theta_0$.

Forced Linear Oscillator

Forced Linear Oscillator: Consider the ODE given by:

$$\ddot{x} + 2\beta\dot{x} + x = \gamma\cos(\omega t), \qquad 0 \le \beta < 1,$$

which can be readily transformed into the ODE system with $x = x_1$ and $\dot{x}_1 = x_2$:

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & -2\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma\cos(\omega t) \end{pmatrix}, \\ \dot{\theta} &= 1. \end{aligned}$$

This system has a forcing function with period $T = 2\pi/\omega$.

One can use techniques from Math 337 (*method of undetermined coefficients*) to solve this problem

$$x(t) = e^{-\beta t}\big(c_1\cos(\omega_d t) + c_2\sin(\omega_d t)\big) + A\cos(\omega t) + B\sin(\omega t),$$

where $\omega_d = \sqrt{1-\omega^2}$ is the damped natural frequency and

$$A = \frac{(1-\omega^2)\gamma}{((1-\omega^2)^2 + 4\beta^2\omega^2)}, \qquad B = \frac{2\beta\omega\gamma}{((1-\omega^2)^2 + 4\beta^2\omega^2)}.$$

Forced Linear Oscillator

Forced Linear Oscillator: The initial conditions determine the c_1 and c_2, so if $x(0) = x_1(0) = x_{10}$ and $\dot{x}(0) = x_2(0) = x_{20}$, then $c_1 = x_{10} - A$ and $c_2 = (x_{20} + \beta(x_{10} - A) - \omega B)/\omega_d$.

Since $\phi_t(x_{10}, x_{20}, 0)$ is given with

$$\begin{aligned} x_1(t) &= e^{-\beta t}\big(c_1\cos(\omega_d t) + c_2\sin(\omega_d t)\big) + A\cos(\omega t) + B\sin(\omega t), \\ x_2(t) &= e^{-\beta t}\Big(-\beta\big(c_1\cos(\omega_d t) + c_2\sin(\omega_d t)\big) + \omega_d\big(-c_1\sin(\omega_d t) + c_2\cos(\omega_d t)\big)\Big) \\ &\quad -\omega\big(A\sin(\omega t) - B\cos(\omega t)\big), \end{aligned}$$

we can compute the *Poincaré map* explicitly as $\Pi[\phi_{2\pi/\omega}(x_{10}, x_20, 0)]$.

This simplifies more in the case of *resonance* when $\omega = \omega_d = \sqrt{1-\beta^2}$, and the *Poincaré map* becomes

$$P(x_{10}, x_{20}, 0) = \begin{pmatrix} (x_{10} - A)e^{-2\pi\beta/\omega} + A \\ (x_{20} - \omega B)e^{-2\pi\beta/\omega} + \omega B \end{pmatrix}.$$

This is readily seen to have a *fixed point* at $(x_1, x_2) = (A, \omega B)$ (when $c_1 = c_2 = 0$).

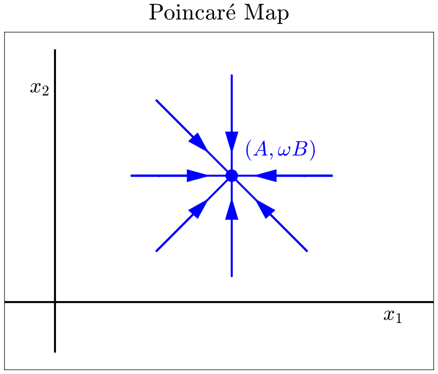
Forced Linear Oscillator

Forced Linear Oscillator: The stability of the *Poincaré map* is determined by the *eigenvalues* of the Jacobian matrix for $P(x_{10}, x_{20}, 0)$

$$\begin{pmatrix} \frac{\partial P_1}{\partial x_{10}} & \frac{\partial P_1}{\partial x_{20}} \\ \frac{\partial P_2}{\partial x_{10}} & \frac{\partial P_2}{\partial x_{20}} \end{pmatrix} = \begin{pmatrix} e^{-2\pi\beta/\omega} & 0 \\ 0 & e^{-2\pi\beta/\omega} \end{pmatrix},$$

which are both eigenvalues with magnitude less than **1**.

Poincaré Map



Method of Averaging

Method of Averaging: We examine some classical methods for problem in nonlinear oscillations.

These techniques build on our studies of *perturbation theory* and extend to studies of *Poincaré maps*.

In a *linear oscillator* problem with *weakly nonlinear* effects or *small perturbations*, one expects that solutions of the *linear oscillator* should be **close** to the *perturbed* problem.

In general, this may **NOT** be the case. However, for *finite time* one usually finds the solutions **close**.

The *method of averaging* is applicable to systems of the form:

$$\dot{x} = \varepsilon f(x,t), \qquad x \in \mathbb{R}^n, \qquad \varepsilon \ll 1,$$

where f is T-periodic in t.

The T-periodic forcing contrasts with the **slow** evolution of the averaged solutions from the $\mathcal{O}(\varepsilon)$ vector field.

Method of Averaging 2

The ***method of averaging*** is applicable to systems of the form:

$$\dot{x} = \varepsilon f(x,t,\varepsilon), \qquad x \in U \subset \mathbb{R}^n, \qquad \varepsilon \ll 1, \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^n$ is C^r, $r \geq 1$, bounded on bounded sets, and T-periodic in t; U is bounded and open.

The associated autonomous averaged system is given by:

$$\dot{y} = \frac{\varepsilon}{T}\int_0^T f(y,t,0)dt = \varepsilon \bar{f}(y). \tag{3}$$

The averaged system (3) should be easier to study, and its properties should reflect the dynamics of (2).

1. A ***weakly nonlinear system*** often has the form
$$\dot{x} = A(t)x + \varepsilon f(x,t,\varepsilon),$$
which doesn't have the form of (2), so how can averaging be applied?
2. Does the ***qualitative behavior*** of the averaged system (3) reflect the behavior of the original system, (2)?

Lagrange Standard Form 1

Consider the ***IVP***:

$$\dot{x} = A(t)x + \varepsilon g(x,t), \qquad x(0) = x_0,$$

where $A(t)$ is a continuous $n \times n$, and $g(x,t)$ is a sufficiently smooth function of t and x.

Assume that $\Phi(t)$ is the **fundamental matrix solution** of the unperturbed system ($\varepsilon = 0$), and $y(t)$ satisfies $y(0) = x_0$ and becomes part of comoving coordinates with

$$x = \Phi(t)y, \qquad \text{so} \qquad \dot{x} = \dot{\Phi}(t)y + \Phi(t)\dot{y}.$$

Since $x(t)$ solves the perturbed system above, we have

$$\dot{\Phi}(t)y + \Phi(t)\dot{y} = A(t)\Phi(t)y + \varepsilon g(\Phi(t)y,t),$$

or

$$\Phi(t)\dot{y} = \big(A(t)\Phi(t) - \dot{\Phi}(t)\big)y + \varepsilon g(\Phi(t)y,t).$$

Lagrange Standard Form 2

Since $\Phi(t)$ is the **fundamental matrix solution** of the unperturbed system, so $\dot{\Phi}(t) = A(t)\Phi(t)$, it follows that:

$$\Phi(t)\dot{y} = \varepsilon g(\Phi(t)y,t), \qquad \text{equivalently} \qquad \dot{y} = \varepsilon \Phi^{-1}(t)g(\Phi(t)y,t).$$

This equation is said to have the ***Lagrange standard form*** and can be written without loss of generality as

$$\dot{y} = \varepsilon f(y,t),$$

which is the same form as our ***weakly nonlinear ODE*** given by (2).

Example of weakly nonlinear forced oscillations: Studies examine:

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x,\dot{x},t),$$

where the ***linear ODE*** with $\varepsilon = 0$ has solutions with

$$x(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t).$$

This 2^{nd} ***order ODE*** is transformed into a 1^{st} ***order system***, then converted to polar coordinates to study the behavior of the periodic solutions.

van der Pol Equation 1

van der Pol Oscillator has been studied for many years due to the interesting behaviors observed, and its behavior simulates a tunnel diode in electric circuits and has been used for simple models of neurons.

The equation is given by

$$\ddot{u} - \varepsilon(1-u^2)\dot{u} + u = 0,$$

where ε is a small parameter.

This equation is readily transformed into the system:

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{pmatrix} v \\ -u + \varepsilon(1-u^2)v\end{pmatrix}. \tag{4}$$

For $\varepsilon = 0$, the solution satisfies:

$$u(t) = r\cos(\theta), \qquad v(t) = -r\sin(\theta),$$

where $\theta = t + \phi$ and the constants r and ϕ are arbitrary representing the ***amplitude*** and ***phase*** of the system.

van der Pol Equation

van der Pol Oscillator: If the ***periodic solution*** of (4) is a continuous function of ε, then the ***orbit*** of this solution should be close to one of the solutions for $\varepsilon = 0$, where r is a constant and θ varies in $[0, 2\pi]$.

We need to find what values of r can generate periodic orbits when $\varepsilon \neq 0$.

Let $r(t)$ and $\theta(t)$ be new coordinates (think polar), then with $u = r\cos(\theta)$ and $v = -r\sin(\theta)$, we have

$$\begin{aligned} \dot{u} &= \dot{r}\cos(\theta) - r\sin(\theta)\dot{\theta}, \\ \dot{v} &= -\dot{r}\sin(\theta) - r\cos(\theta)\dot{\theta}. \end{aligned}$$

It is not hard to see that this gives

$$\begin{aligned} \dot{r} &= \dot{u}\cos(\theta) - \dot{v}\sin(\theta), \\ r\dot{\theta} &= -\dot{u}\sin(\theta) - \dot{v}\cos(\theta). \end{aligned}$$

However, we know $\dot{u}$ and $\dot{v}$ from (4), so we can insert them into the equation above.

van der Pol Equation

van der Pol Oscillator: With the substitutions and a little algebra we obtain the new system in the transformed coordinates:

$$\begin{aligned} \dot{\theta} &= 1 + \varepsilon\big(1 - r^2\cos^2(\theta)\big)\sin(\theta)\cos(\theta), \\ \dot{r} &= \varepsilon\big(1 - r^2\cos^2(\theta)\big)r\sin^2(\theta). \end{aligned} \tag{5}$$

For ε chosen such that $1 + \varepsilon\big(1 - r^2\cos^2(\theta)\big)\sin(\theta)\cos(\theta) > 0$ and r in a bounded set, then the ***orbits*** are described by the solutions of the scalar equation:

$$\frac{dr}{d\theta} = \varepsilon g(r, \theta, \varepsilon), \tag{6}$$

where

$$g(r, \theta, \varepsilon) = \frac{\big(1 - r^2\cos^2(\theta)\big)r\sin^2(\theta)}{1 + \varepsilon\big(1 - r^2\cos^2(\theta)\big)\sin(\theta)\cos(\theta)}.$$

This reduces finding periodic solutions of ***van der Pol's equation*** to finding periodic solutions of the scalar equation (6) of period 2π.

van der Pol Equation

van der Pol Oscillator: We seek to find periodic solutions $r^*(\theta, \varepsilon)$ of (6) of period 2π in θ.

In fact, if $r^*(\theta, \varepsilon)$ is such a 2π-periodic solution and $\theta^*(t, \varepsilon)$, $\theta^*(0, \varepsilon) = 0$ solves the equation:

$$\dot{\theta} = 1 + \varepsilon\big(1 - [r^*(\theta, \varepsilon)]^2\cos^2(\theta)\big)\sin(\theta)\cos(\theta),$$

then

$$u(t) = r^*(\theta^*(t, \varepsilon), \varepsilon)\cos(\theta^*(t, \varepsilon)), \qquad v(t) = -r^*(\theta^*(t, \varepsilon), \varepsilon)\sin(\theta^*(t, \varepsilon)),$$

is a solution of ***van der Pol's equation***.

Let T be the unique solution of $\theta^*(T, \varepsilon) = 2\pi$. Then uniqueness of the $\dot{\theta}$ equation implies $\theta^*(t + T, \varepsilon) = \theta^*(t, \varepsilon) + 2\pi$ for all t.

Thus, $u(t + T) = u(t)$, $v(t + T) = v(t)$ giving a T-periodic solution to ***van der Pol's equation***.

We see that solving (6),

$$\frac{dr}{d\theta} = \varepsilon g(r, \theta, \varepsilon),$$

fits into our studies of ***perturbation problems***.

Method of Averaging Theorem

The ***method of averaging*** is applicable to systems of the form:

$$\dot{x} = \varepsilon f(x, t, \varepsilon), \qquad x \in U \subset \mathbb{R}^n, \qquad \varepsilon \ll 1.$$

Theorem (The Averaging Theorem)

There exists a C^r change of coordinates $x = y + \varepsilon w(y, t, \varepsilon)$ under which (2) becomes

$$\dot{y} = \varepsilon \bar{f}(y) + \varepsilon^2 f_1(y, t, \varepsilon),$$

where f_1 is of period T in t. Moreover,

1. *If $x(t)$ and $y(t)$ are solutions of (2) and (3) based at x_0, y_0, respectively, at $t = 0$, and $|x_0 - y_0| = \mathcal{O}(\varepsilon)$, then $|x(t) - y(t)| = \mathcal{O}(\varepsilon)$ on a time scale $t \sim \frac{1}{\varepsilon}$.*
2. *If p_0 is a hyperbolic fixed point of (3) then there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_0$, (2) possesses a unique hyperbolic periodic orbit $\gamma_\varepsilon(t) = p_0 + \mathcal{O}(\varepsilon)$ of the same stability type as p_0.*
3. *If $x^s(t) \in W^s(\gamma_\varepsilon)$ is a solution of (2) lying in the stable manifold of the hyperbolic periodic orbit $\gamma_\varepsilon = p_0 + \mathcal{O}(\varepsilon)$, $y^s(t) \in W^s(p_0)$ is a solution of (3) lying in the stable manifold of the hyperbolic fixed point p_0 and $|x^s(0) - y^s(0)| = \mathcal{O}(\varepsilon)$, then $|x^s(t) - y^s(t)| = \mathcal{O}(\varepsilon)$ for $t[0, \infty)$. Similar results apply to solutions lying in the unstable manifolds on the time interval $t \in (-\infty, 0]$.*

van der Pol - revisited

van der Pol Oscillator: We examine the more general problem:

$$\ddot{u} + u = \varepsilon F(u, \dot{u}, t),$$

where for the ***van der Pol oscillator*** $F(u, \dot{u}, t) = -(u^2 - 1)\dot{u}$.

We attempt a solution of the form:

$$u(t) = r(t)\cos(t + \theta(t)), \qquad \dot{u} = -r(t)\sin(t + \theta(t)),$$

motivated by the idea that r and θ are constants when $\varepsilon = 0$ and the functions $r(t)$, ***amplitude***, and $\theta(t)$, ***phase***, are slow varying functions of t.

Differentiating $u(t)$ and requiring the second to hold gives:

$$\dot{r}\cos(t + \theta(t)) - r\dot{\theta}\sin(t + \theta(t)) = 0.$$

Finding $\ddot{u}$ gives:

$$-\dot{r}\sin(t + \theta(t)) - r\dot{\theta}\cos(t + \theta(t)) = \varepsilon F\big(r(t)\cos(t + \theta(t)), -r(t)\sin(t + \theta(t)), t\big).$$

van der Pol - revisited

van der Pol Oscillator: The equations above are solved to give the ***generalized system in amplitude and phase***:

$$\begin{aligned} \dot{r} &= \varepsilon - F\big(r\cos(t+\theta), -r\sin(t+\theta), t\big)\sin(t+\theta), \\ \dot{\theta} &= -\frac{\varepsilon}{r}F\big(r\cos(t+\theta), -r\sin(t+\theta), t\big)\cos(t+\theta). \end{aligned}$$

For ε small and θ constant, this system would satisfy our ***Method of Averaging Theorem***. However, $\theta(t)$ is slow varying, so the above system is not quite ***2π-periodic***.

Introduce an approximation, using a ***near-identity transformation***:

$$r(t) = \bar{r} + \varepsilon w_1(\bar{r}, \bar{\theta}, \varepsilon) + \mathcal{O}\left(\varepsilon^2\right), \qquad \theta(t) = \bar{\theta} + \varepsilon w_2(\bar{r}, \bar{\theta}, \varepsilon) + \mathcal{O}\left(\varepsilon^2\right),$$

where w_1 and w_2 are ***generating functions*** such that $\bar{r}$ and $\bar{\theta}$ are as simple as possible.

This gives the approximations:

$$\begin{aligned} \frac{d\bar{r}}{dt} &= \varepsilon\left(-\frac{\partial w_1}{\partial t} - \sin(t+\bar{\theta})F\big(\bar{r}\cos(t+\bar{\theta}), -\bar{r}\sin(t+\bar{\theta}), t\big)\right) + \mathcal{O}\left(\varepsilon^2\right), \\ \frac{d\bar{\theta}}{dt} &= \varepsilon\left(-\frac{\partial w_2}{\partial t} - \frac{\cos(t+\bar{\theta})}{\bar{r}}F\big(\bar{r}\cos(t+\bar{\theta}), -\bar{r}\sin(t+\bar{\theta}), t\big)\right) + \mathcal{O}\left(\varepsilon^2\right). \end{aligned}$$

van der Pol - revisited

van der Pol Oscillator: To avoid having ***secular terms*** we choose w_1 and w_2 to eliminate all $\mathcal{O}(\varepsilon)$ terms except for their average value.

The ***averaged equations*** become:

$$\begin{aligned} \frac{d\bar{r}}{dt} &= -\varepsilon\frac{1}{T}\int_0^T \sin(t+\bar{\theta})F\big(\bar{r}\cos(t+\bar{\theta}), -\bar{r}\sin(t+\bar{\theta}), t\big)dt + \mathcal{O}\left(\varepsilon^2\right), \\ \frac{d\bar{\theta}}{dt} &= -\varepsilon\frac{1}{T}\int_0^T \frac{\cos(t+\bar{\theta})}{\bar{r}}F\big(\bar{r}\cos(t+\bar{\theta}), -\bar{r}\sin(t+\bar{\theta}), t\big)dt + \mathcal{O}\left(\varepsilon^2\right). \end{aligned}$$

For the ***autonomous ODE***, the averaging period is $T = 2\pi$ and these equations reduce to the form:

$$\begin{aligned} \frac{d\bar{r}}{dt} &= -\varepsilon\frac{1}{2\pi}\int_0^{2\pi} \sin(t)F\big(\bar{r}\cos(t), -\bar{r}\sin(t)\big)dt + \mathcal{O}\left(\varepsilon^2\right), \\ \frac{d\bar{\theta}}{dt} &= -\varepsilon\frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(t)}{\bar{r}}F\big(\bar{r}\cos(t), -\bar{r}\sin(t)\big)dt + \mathcal{O}\left(\varepsilon^2\right), \end{aligned}$$

where we see that the ***slow amplitude*** variation ODE is decoupled.

van der Pol - revisited

Many derivations of the ***van der Pol oscillator*** omit the ***near-identity transformation***.

Knowing this transformation allows greater accuracy in transforming back to the original variables r and θ, and secondly, one can obtain ***higher order approximations*** by simply extending our approximations above to $\mathcal{O}\left(\varepsilon^3\right)$.

van der Pol Oscillator: Now consider

$$F(u, \dot{u}, t) = (1 - u^2)\dot{u},$$

then the averaged equation becomes:

$$\begin{aligned} \frac{d\bar{r}}{dt} &= \varepsilon\frac{1}{2\pi}\int_0^{2\pi} \bar{r}\sin^2(t)\big(1 - \bar{r}^2\cos^2(t)\big)dt + \mathcal{O}\left(\varepsilon^2\right), \\ \frac{d\bar{\theta}}{dt} &= \varepsilon\frac{1}{2\pi}\int_0^{2\pi} \cos(t)\sin(t)\big(1 - \bar{r}^2\cos^2(t)\big)dt + \mathcal{O}\left(\varepsilon^2\right), \end{aligned}$$

where we see that the ***slow amplitude*** variation ODE is decoupled.

van der Pol - revisited

van der Pol Oscillator: Omitting the $\mathcal{O}(\varepsilon^2)$, the averaged equation is easily integrated:

$$\frac{d\bar{r}}{dt} = \varepsilon\frac{1}{2\pi}\int_0^{2\pi}\bar{r}\sin^2(t)\big(1-\bar{r}^2\cos^2(t)\big)dt = \varepsilon\frac{\bar{r}}{8}(4-\bar{r}^2),$$

$$\frac{d\bar{\theta}}{dt} = \varepsilon\frac{1}{2\pi}\int_0^{2\pi}\cos(t)\sin(t)\big(1-\bar{r}^2\cos^2(t)\big)dt = 0.$$

The ***nonlinear ODE*** in $\bar{r}$ can be analyzed ***qualitatively***.

It has ***two negative equilibria,*** $\bar{r}_e = 0, 2$.

The ***equilibrium*** at $\bar{r}_e = 0$ has a ***positive eigenvalue***, so it results in an ***unstable node*** with solutions spiraling away from the origin.

The ***equilibrium*** at $\bar{r}_e = 2$ has a ***negative eigenvalue***, so it results in an ***stable node***, which corresponds to a stable almost 2π-periodic orbit of radius 2.

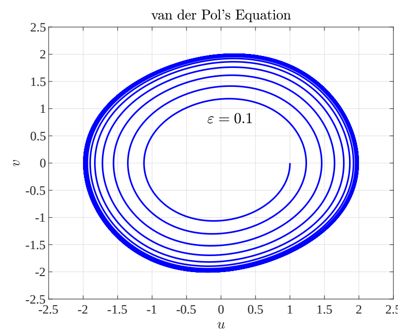
The ***ODE*** for $\bar{\theta}$ shows that up to $\mathcal{O}(\varepsilon^2)$ the phase shift remains constant.

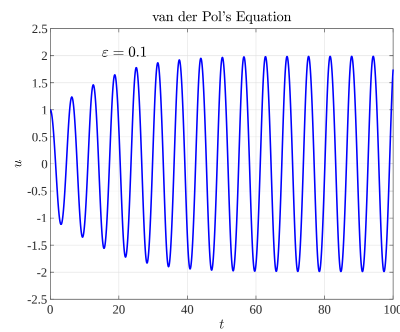
van der Pol - revisited

van der Pol Oscillator: The averaged equation for $\bar{r}$ can be solved exactly by separation of variables and gives the result:

$$\bar{r}(t) = \frac{2e^{\varepsilon t/2}}{\sqrt{e^{\varepsilon t} - 1 + \frac{4}{\bar{r}(0)^2}}}.$$

Below are graphs for the ***van der Pol oscillator*** for small ε.





van der Pol - revisited

Below are graphs for the ***van der Pol oscillator*** for large ε. These show why this is often called a ***relaxation oscillator***.