Math 537 - Ordinary Differential Equations

Lecture Notes – Linear Differential Equations and Scaling

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Outline

Carbon Radiodating

Carbon Radiodating: One important application of radioactive decay is the dating of biological specimens.

- A living organism is continually changing its carbon with the environment.
  - Plants directly absorb CO_2 from the atmosphere.
  - Animals get their carbon either directly or indirectly from plants.
- Gamma radiation that bombards the Earth keeps the ratio of ^{14}C to ^{12}C fairly constant in the atmospheric CO_2.
  - Atomic tests in 1950s and 1960s increased gamma radiation.
  - Large releases of CO_2 (global warming) are making the dating less reliable.
- Until recently, ^{14}C stays at a constant concentration until the organism dies.

Carbon Radiodating

Modeling Carbon Radiodating: Radioactive carbon, ^{14}C, decays with a **half-life of 5730 yr.**

- Living tissue shows a radioactivity of about 15.3 disintegrations per minute (dpm) per gram of carbon.
- The loss of ^{14}C from a sample at any time t is proportional to the amount of ^{14}C remaining.
- Let $R(t)$ be the dpm per gram of ^{14}C from an ancient object.
- The differential equation for a gram of ^{14}C

$$\frac{dR(t)}{dt} = -kR(t) \qquad \text{with} \qquad R(0) = 15.3.$$

- This differential equation has the solution:

$$R(t) = 15.3\, e^{-kt}, \quad \text{where} \quad k = \frac{\ln(2)}{5730} = 0.000121.$$

Example: Carbon Radiodating

Example Carbon Radiodating: Suppose that an object is found to have a radioactive count of 5.2 dpm per g of carbon

Find the age of this object.

Solution: From above

$$\begin{aligned} 5.2 &= 15.3\, e^{-kt} \qquad \text{or} \qquad e^{kt} = \tfrac{15.3}{5.2} = 2.94. \\ kt &= \ln(2.94) \qquad \text{with} \qquad k = \tfrac{\ln(2)}{5730} = 0.000121. \end{aligned}$$

Thus, $t = \frac{\ln(2.94)}{k} = 8915$ yr, so the object is about 9000 yrs old.

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Art Forgery

Van Meegeren Art Forgery: At the end of WWII, H. A. Van Meegeren was arrested for collaborating with the Germans in the sale of the painting "Woman Taken in Adultery" by Jan Vermeer to Goering.[1]

- Van Meegeren was considered a 3^{rd} rate painter.
- From prison to avoid charges of treason announced he forged this painting and a number of other famous paintings.
- Several of the paintings were extremely well-done, so several art experts didn't believe him.
- He was just about to show how he created the masterpieces, including his technique to age the paintings, when charges were changed to forgery.
- Some of the paintings were easily shown to be forgeries, so he was convicted and sent to prison.
- However, his "Disciples of Emmaus" was so good, it fooled experts and was certified genuine.
- Scientists used the radioactive ^{210}Pb from lead oxide (white lead) to prove it was a fake.

1. M. Braun, *Differential Equations and Their Applications*, Springer-Verlag, 1983.

Radioactive Cascade

Radioactive Lead ^{210}Pb: One source of the stable element lead, ^{206}Pb, is through a series of decaying radioactive elements starting with uranium, ^{238}U (half-life 4.5 billion years), and ending with lead, ^{206}Pb.

We focus on the cascade of elements from radium, ^{226}Ra to lead, ^{206}Pb.

$$^{226}\text{Ra} \overset{h_1}{\longrightarrow} {}^{222}\text{Rn} \overset{h_2}{\longrightarrow} {}^{218}\text{Po} \overset{h_3}{\longrightarrow} {}^{214}\text{Pb} \overset{h_4}{\longrightarrow} {}^{214}\text{Bi}$$

$$\overset{h_5}{\longrightarrow} {}^{214}\text{Po} \overset{h_6}{\longrightarrow} {}^{210}\text{Pb} \overset{h_7}{\longrightarrow} {}^{210}\text{Bi} \overset{h_8}{\longrightarrow} {}^{210}\text{Po} \overset{h_9}{\longrightarrow} {}^{206}\text{Pb}$$

The half-lives are given by h_i with

$h_1 = 1600$ yr	$h_2 = 3.82$ da	$h_3 = 3.05$ min	$h_4 = 26.8$ min	$h_5 = 19.7$ min
$h_6 = 0.16$ msec	$h_7 = 22$ yr	$h_8 = 5.0$ da	$h_9 = 138$ da	-

- This ***radioactive cascade*** system can be formulated into a system of ordinary differential equations. This course will study **large linear systems of ODEs**.
- The decay rates span a wide range, and this course will examine how to manage some **multi-scale problems**.
- Often try to connect theory to practical problems.

Radioactive ODE System

ODE System: Define x_0 as the amount of ^{226}Ra, x_1 as the amount of ^{222}Rn, x_2 as the amount of ^{218}Po, etc.

Define the decay rates $k_i = \frac{\ln(2)}{h_i}$, $\quad i = 1, ..., 9$.

The ***linear nonhomogeneous system of ODEs*** satisfy:

$$\dot{x}_i = k_{i-1}x_{i-1} - k_i x_i, \qquad i = 1, ..., 9,$$

where $k_0 x_0$ is the rate of decay of ^{226}Ra $\times$ amount of ^{226}Ra.

With $\mathbf{x} = [x_1, x_2, ..., x_9]^T$, this can be written as the **ODE system**:

$$\dot{\mathbf{x}} = A\mathbf{x} + B,$$

where

$$A = \begin{pmatrix} -k_1 & 0 & \dots & \dots & 0 \\ k_1 & -k_2 & 0 & \dots & 0 \\ 0 & k_2 & -k_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & k_8 & -k_9 \end{pmatrix} \qquad B = \begin{pmatrix} k_0 x_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Radioactive – Quasi-steady state

Half-lives: The earlier table shows the half-lives in this radioactive cascade varying from $h_1 = 1600$ yr to $h_6 = 0.16$ msec, which is an extremely wide range of time scales.

When time scales vary by orders of magnitude, one often invokes a ***quasi-steady state*** assumption that the specific equation is so fast that it is effectively in a temporary steady state or ***equilibrium***, so $\dot{x}_i = 0$.

This reduces the dimension of the ODE by creating an algebraic equation:

$$\dot{x}_i = 0 = k_{i-1}x_{i-1} - k_i x_i \qquad \text{or} \qquad k_{i-1}x_{i-1} = k_i x_i,$$

which can be substituted into the ***system of ODEs***.

This course will examine **slow** and **fast time scales**, **scaling of variables**, and possibly study **multi-time scale problems**.

Art Forgery

Van Meegeren Art Forgery: White lead has been used in paintings for over 2000 years.

- The smelting process to obtain lead removes much of the radium, which decays to ^{210}Pb.
- The method to detect forgeries looks at the ratio of ^{210}Pb to ^{226}Ra.
- An authentic Vermeer would have an age of over 300 yrs, while the Van Meegeren forgeries would be fairly recent.
- With these time scales we can assume that the ^{226}Ra has a roughly constant amount.
- The other intermediate elements are on a fast time scale allowing *quasi-steady state* approximations.
- If $y(t)$ is the amount of ^{210}Pb, then these assumptions result in the **scalar ODE**:

$$\dot{y} = r - ky, \qquad \text{where} \quad k = \frac{\ln(2)}{22}.$$

Linear ODE

Radioactive Decay: The **ODE** for a radioactive substance, y, which has a constant source, satisfies:

$$\dot{y} + ky = r.$$

This **ODE** is

- **Linear**, as the ***dependent variable***, y, and its ***derivative*** only appears linearly.
- **First Order**, as the equation only has the ***first derivative***.
- **Non-homogeneous**, as the constant, r, appears on the rhs.

We use operator notation:

$$\mathcal{L}[y] = r, \qquad \text{where} \quad \mathcal{L}[y] = \left(\frac{d}{dt} + k\right) y.$$

Linear Operator

Linear Operator: We show that $\mathcal{L}$ is a linear operator.

$$\begin{aligned}
\mathcal{L}[\alpha y_1 + \beta y_2] &= \left(\frac{d}{dt} + k\right)[\alpha y_1 + \beta y_2] \\
&= \frac{d}{dt}(\alpha y_1 + \beta y_2) + k(\alpha y_1 + \beta y_2) \\
&= \alpha \frac{dy_1}{dt} + \beta \frac{dy_2}{dt} + \alpha k y_1 + \beta k y_2 \\
&= \alpha\left(\frac{dy_1}{dt} + k y_1\right) + \beta\left(\frac{dy_2}{dt} + k y_2\right) \\
&= \alpha \mathcal{L}[y_1] + \beta \mathcal{L}[y_2]
\end{aligned}$$

Homogeneous problem is $\mathcal{L}[y] = 0$.

Non-homogeneous problem is $\mathcal{L}[y] = r(t)$.

Initial Value Problem

Solution of Initial Value Problem: Consider the problem:

$$\dot{y} + ky = r, \qquad y(t_0) = y_0.$$

This problem is readily solved with an ***integrating factor***.

By multiplying the equation above by $\mu(t) = e^{kt}$, the left hand side becomes an exact differential:

$$e^{kt}\left(\frac{dy}{dt} + ky\right) = \frac{d}{dt}\left(e^{kt}y(t)\right) = re^{kt}.$$

Integrating produces

$$\int_{t_0}^{t} \frac{d}{ds}\left(e^{ks}y(s)\right)ds = \int_{t_0}^{t} re^{ks}ds,$$

or

$$e^{kt}y(t) - e^{kt_0}y_0 = \frac{r}{k}\left(e^{kt} - e^{kt_0}\right).$$

Initial Value Problem

Solution of Initial Value Problem: It follows that the solution of

$$\dot{y} + ky = r, \qquad y(t_0) = y_0,$$

is

$$y(t) = y_0 e^{-k(t-t_0)} + \frac{r}{k}\left(1 - e^{-k(t-t_0)}\right).$$

Equilibrium: Note the equilibrium occurs when $\dot{y} = 0$, so from the **ODE** we see $ky_e = r$ or

$$y_e = \frac{r}{k}.$$

From the solution above, we see:

$$\lim_{t\to\infty} y(t) = \frac{r}{k}.$$

It follows that $y_e = \frac{r}{k}$ is an ***asymptotically stable equilibrium*** with all solutions approaching y_e for large time.

We also note that the homogeneous **ODE** has the ***eigenvalue***, $\lambda = -k < 0$.

Art Forgery

Van Meegeren Art Forgery: Experts believed the "Disciples of Emmaus" was so good that it must be a real Jan Vermeer.

Samples of the white paint with white lead were analyzed for ^{210}Pb and ^{226}Ra.

As a surrogate for ^{210}Pb, ^{210}Po was measured at 8.5 distingrations/min (***quasi-steady state***).

The ^{226}Ra decay was 0.8 distingrations/min.

These data translate into $ky(t) = 8.5$ and $r = 0.8$ when measured.

Find ky_0 when the pigment was formulated.

If this was a real Vermeer, then $t - t_0 \approx 300$ yr.

Art Forgery

Van Meegeren Art Forgery: From the solution of the radioactive decay problem, the data from the white lead measurements, and the half-life of ^{210}Pb, we have:

$$ky(t) = 8.5 \approx ky_0 e^{-300k} + r\left(1 - e^{-300k}\right),$$

where $k = \frac{\ln(2)}{22} \approx 0.0315$ and $r = 0.8$.

It follows that

$$ky_0 \approx (8.5 - 0.8)e^{300k} + 0.8 \approx 98,049 \text{ disintegrations/min}.$$

The richest known sources of ^{226}Ra have only a few thousand disintegrations/min.

It follows that the source of the white pigment could not have been refined 300 yr ago, so the painting is significantly more recent.

This painting was a Van Meegeren forgery created in the 1930s.

Linear Differential Equation 1

Consider the **Linear Differential Equation**:

$$y' + p(t)y = g(t), \qquad \text{with} \qquad y(t_0) = y_0. \tag{1}$$

Assume p and g are continuous on an open interval $I : \alpha < t < \beta$ with $t_0 \in (\alpha, \beta)$, so p and g are integrable on I.

Definition (Integrating Factor)

Consider an undetermined function $\mu(t)$ with

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y.$$

The function $\mu(t)$ is an **integrating factor** for (1) if it satisfies the differential equation

$$\frac{d\mu(t)}{dt} = p(t)\mu(t).$$

Linear Differential Equation 2

The differential equation for the **integrating factor** is

$$\frac{d\mu(t)}{dt} = p(t)\mu(t) \qquad \text{or} \qquad \frac{1}{\mu(t)}\frac{d\mu(t)}{dt} = p(t).$$

Note that $\frac{d(\ln(\mu(t)))}{dt} = \frac{1}{\mu(t)}\frac{d\mu(t)}{dt}$.

From the ***Fundamental Theorem of Calculus***:

$$\ln(\mu(t)) - \ln(\mu(t_0)) = \int_{t_0}^{t} p(s)ds.$$

It follows that the **general integrating factor** satisfies

$$\mu(t) = e^{\int_{t_0}^{t} p(s)ds}.$$

(Note: $\ln(\mu(t_0)) = 0$, as $\mu(t_0) = 1$.)

Linear Differential Equation 3

Multiplying the **Linear Differential Equation** by $\mu(t)$ gives:

$$\mu(t)\left(y' + p(t)y\right) = \frac{d}{dt}\left[\mu(t)y\right] = \mu(t)g(t),$$

which upon integration gives:

$$\mu(t)y(t) - y_0 = \int_{t_0}^{t} \mu(s)g(s)ds.$$

It follows that the **unique solution** to the ***linear ODE*** (1) is

$$y(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)ds + y_0\right),$$

where

$$\mu(t) = e^{\int_{t_0}^{t} p(s)ds}.$$

Linear Differential Equation 4

The **linear ODE** has the following existence and uniqueness result.

Theorem

If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing a point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in I with the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

Linear Differential Equation

The summary result for the **linear ODE** is the following.

Theorem (Solution of 1^{st} Order Linear DE)

Assume the 1^{st} order linear ODE given by (1) and the conditions on p and g of the previous theorem. Then there exists a unique solution for $t \in (\alpha, \beta)$ given by

$$y(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^{t} \mu(s) g(s) ds + y_0 \right),$$

where

$$\mu(t) = e^{\int_{t_0}^{t} p(s) ds}.$$

We note that the existence and uniqueness result for the 1^{st} ***order nonlinear ODE*** is more complicated and may be examined later in the course.

Falling Object with Drag 1

Example: Consider an object falling under the force of gravity with resistance linearly increasing in time and proportional to the velocity of the object.

By Newton's Law we can write this as a balance of forces:

$$m\frac{dv}{dt} = -ktv - mg,$$

which when divided by the mass, m, and defining $\alpha = \frac{k}{m}$ becomes the ***linear ODE***:

$$\frac{dv}{dt} + \alpha tv = -g, \qquad \text{with} \quad v(0) = v_0.$$

The integrating factor is given by:

$$\mu(t) = e^{\int_0^t \alpha s\, ds} = e^{\alpha t^2/2}.$$

Falling Object with Drag

Example: With the integrating factor, $\mu(t) = e^{\alpha t^2/2}$, the solution becomes:

$$\begin{aligned} v(t) &= v_0 e^{-\alpha t^2/2} + e^{-\alpha t^2/2} \int_0^t (-g) e^{\alpha s^2/2} ds, \\ v(t) &= v_0 e^{-\alpha t^2/2} - g\, e^{-\alpha t^2/2} \int_0^t e^{\alpha s^2/2} ds. \end{aligned}$$

What is the limiting velocity for this model?

The first term, $v_0 e^{-\alpha t^2/2}$, clearly tends to **zero** as $t \to \infty$.

However, what about

$$\lim_{t\to\infty} e^{-\alpha t^2/2} \int_0^t e^{\alpha s^2/2} ds = \lim_{t\to\infty} \frac{\int_0^t e^{\alpha s^2/2} ds}{e^{\alpha t^2/2}}?$$

Falling Object with Drag 3

Example: Since $e^{\alpha s^2/2} > 1$, the integral is unbounded, which implies both numerator and denominator tend to $+\infty$.

So we apply **L'Hôspital's Rule**, which gives:

$$\lim_{t\to\infty} \frac{\int_0^t e^{\alpha s^2/2} ds}{e^{\alpha t^2/2}} = \lim_{t\to\infty} \frac{e^{\alpha t^2/2}}{\alpha t\, e^{\alpha t^2/2}} = \lim_{t\to\infty} \frac{1}{\alpha t} = 0.$$

It follows that:

$$\lim_{t\to\infty} v(t) = 0,$$

so the falling object would eventually have sufficient drag to halt the object.

Parameters and Scaling

Parameters: Crucial to any modeling problem are the parameters.

- These often describe important properties of the system.
- Parameters are often fit to data.
- Variations in parameters may have a critical role in the dynamical behavior.
- Questions arise on what is called ***identifiability*** of the parameters.
- More parameters can complicate the understanding of key underlying behaviors.

Buckingham Pi Theorem

Theorem (Buckingham Pi Theorem)

Let q_1, q_2, q_3, ..., q_n be n dimensional variables that are physically relevant in a given problem and that are inter-related by an (unknown) dimensionally homogeneous set of equations. These can be expressed via a functional relationship of the form:

$$F(q_1, q_2, ...q_n) = 0 \qquad \text{or equivalently} \qquad q_1 = f(q_2, ...q_n).$$

If k is the number of fundamental dimensions required to describe the n variables, then there will be k primary variables and the remaining variables can be expressed as $(n-k)$ dimensionless and independent quantities or Pi groups, $\Pi_1, \Pi_2, ..., \Pi_{n-k}$. The functional relationship can tbe reduced to the much more compact form:

$$\Phi(\Pi_1, \Pi_2, , \Pi_{n-k}) = 0 \qquad \text{or equivalently} \qquad \Pi_1 = \Phi(\Pi_2, , \Pi_{n-k}).$$

Rayleigh's Method of Dimensional Analysis

Rayleigh's method of dimensional analysis

- Gather all the independent variables that are likely to influence the dependent variable.
- If R is a variable that depends upon independent variables $R_1, R_2, R_3, ..., R_n$, then the functional equation can be written as $R = F(R_1, R_2, R_3, ..., R_n)$.
- Write the above equation in the form $R = CR_1^a R_2^b R_3^c ... R_n^m$, where C is a dimensionless constant and $a, b, c, ..., m$ are arbitrary exponents.
- Express each of the quantities in the equation in some base units in which the solution is required.
- By using dimensional homogeneity, obtain a set of simultaneous equations involving the exponents $a, b, c, ..., m$.
- Solve these equations to obtain the value of exponents $a, b, c, ..., m$.
- Substitute the values of exponents in the main equation, and form the non-dimensional parameters by grouping the variables with like exponents.

Dimensional Analysis

Dimensional Analysis – Primary Units

There are a number of primary units:

Length	Mass	Time	Amount	Temperature	Electricity	Luminosity
L	M	T	N	Q	I	C

Example 1: Newton's Law of Force is given by

$$F = ma$$

This could be written

$$\frac{F}{ma} - 1 = 0,$$

which gives the ***dimensionless quantity***

$$\Pi = \frac{F}{ma}, \qquad \text{so} \qquad f(\Pi) = \Pi - 1.$$

Dimensional Analysis – Example 1

Example - Launching: Consider launching an object with critical quantities: $m =$ mass, $v =$ launch velocity, $h =$ maximum height, and $g =$ acceleration gravity

Choose:

$$[m] = M \qquad [v] = LT^{-1} \qquad [h] = L \qquad [g] = LT^{-2}$$

Create the ***dimensionless quantity***:

$$\Pi = m^a v^b h^c g^d$$

Analyze the exponents for quantities M, L, and T, so to be dimensionless

$$a = 0 \qquad b + c + d = 0 \qquad -b - 2d = 0.$$

Dimensional Analysis – Example 2

Example (cont): There are **4** coefficients a, b, c, and d for the **3** dimensional variables M, L, and T, leaving one free parameter.

With the one degree of freedom, we take $d = c$ and $c = 1$, then the coefficients become

$$a = 0 \qquad b = -2 \qquad c = 1 \qquad d = 1.$$

The dimensionless variable is

$$\Pi = \frac{hg}{v^2} \qquad f(\Pi) = f\left(\frac{hg}{v^2}\right) = 0.$$

It follows that

$$\frac{hg}{v^2} = k \qquad \text{or} \qquad h = \frac{kv^2}{g}.$$

Dimensional Analysis – Example 3

Example (cont): Since

$$h = \frac{kv^2}{g},$$

it follows that the ***height of a launch*** depends only on the quantity v^2/g.

- The ***height of a launch*** is independent of the ***mass***.
- The ***height of a launch*** varies as the square of the ***velocity***.
- The ***height of a launch*** is inversely proportional to the acceleration of ***gravity***.

It follows that doubling the launch velocity increases the height of the launch by a factor of 4.
On the moon with gravity, $\frac{g}{6}$, the height of the launch increases by a factor of 6.

Dimensional Analysis – Atomic Bomb 1

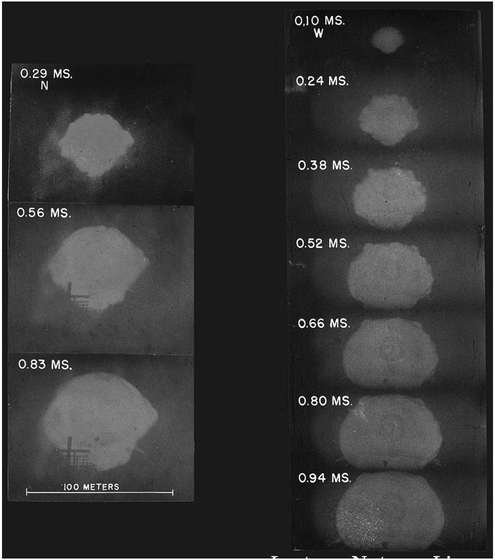
Example – Atomic Bomb: Sir Geoffrey Taylor F.R.S., The formation of a blast wave by a very intense explosion: II. The atomic explosion of 1945, *Proc. R. Soc. Lond.*, **A**, (1950)

This article used a movie of the Trinity test with dimensional analysis to estimate the power of the explosion.

Pictures of the White Sands, NM test in 1945 showed the radius of explosion:

Time, t (sec)	Blast Radius, R (m)	Time, t (sec)	Blast Radius, R (m)
0.00038	25.4	0.0008	34.2
0.00052	28.8	0.00094	36.3
0.00066	31.9	0.00108	38.9

Dimensional Analysis – Atomic Bomb 2



Dimensional Analysis – Atomic Bomb 3

Atomic Bomb (cont): Assume that the radius, R, of Atomic blast depends only on time, t, ambient density, ρ, and Energy, E, of the explosion – we ignore other effects.

From the Buckingham Pi Theorem, the ***dimensionless variable*** satisfies:

$$\Pi = R^a E^b t^c \rho^d,$$

where

$$[R] = L \qquad [E] = \frac{ML^2}{T^2} \qquad [t] = T \qquad [\rho] = \frac{M}{L^3},$$

so

$$\Pi = L^a \left(\frac{ML^2}{T^2}\right)^b T^c \left(\frac{M}{L^3}\right)^d.$$

Dimensional Analysis – Atomic Bomb 4

Atomic Bomb (cont): From before, the ***dimensionless variable*** satisfies:

$$\Pi = L^a \left(\frac{ML^2}{T^2}\right)^b T^c \left(\frac{M}{L^3}\right)^d.$$

From the coefficients above we have

$$\begin{aligned} a + 2b - 3d &= 0 \quad (L) \\ b + d &= 0 \quad (M) \\ -2b + c &= 0 \quad (T) \end{aligned}$$

There is one degree of freedom, so let $b = 1$, then

$$a = -5 \qquad b = 1 \qquad c = 2 \qquad d = -1.$$

Dimensional Analysis – Atomic Bomb

Atomic Bomb (cont): From the ***dimensionless variable***, we write

$$\Pi = R^{-5}Et^2\rho^{-1} \qquad \text{or} \qquad R = k\left(\frac{Et^2}{\rho}\right)^{1/5}.$$

The Taylor article goes to some length to show that $k \approx 1$ and $\rho \approx 1$.

Air has $\rho = 1.2$ kg/m^3 at sea level, and White sands is at 1200 m, which has a density of 1.03 kg/m^3

It follows that

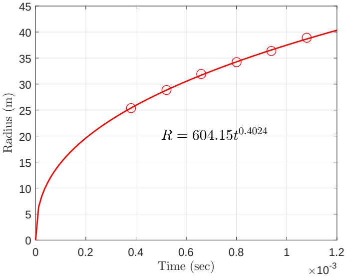
$$R = (Et^2)^{1/5},$$

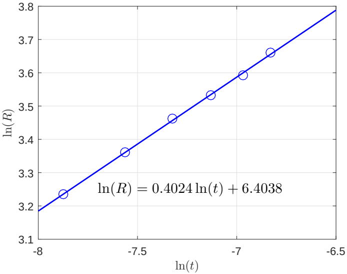
which is a ***power law*** or ***allometric model*** and

$$\ln(R) = \frac{1}{5}\ln(E) + \frac{2}{5}\ln(t).$$

Dimensional Analysis – Atomic Bomb

Below are graphs of the data and the ln of the data:





Dimensional Analysis – Atomic Bomb 7

From before we have the ***power law*** or ***allometric model***

$$R = (Et^2)^{1/5} \qquad \text{or} \qquad \ln(R) = \frac{1}{5}\ln(E) + \frac{2}{5}\ln(t),$$

and the slope of the logarithmic graph from the data agrees with the coefficient obtained by ***dimensional analysis***.

From the data we obtain the intercept, so

$$\frac{1}{5}\ln(E) = 6.4038,$$

which is readily solved for E giving the energy of the atomic blast as

$$E = e^{32.02} = 8.05 \times 10^{13} \text{ J}.$$

Scientists running experiments at the blast site measured the power of the trinity atomic blast as 9×10^{13} J.

Scaling Parameters in ODE 1

Consider the **ODE** given by:

$$\frac{dy}{dt} + \alpha p(\omega t) y = \beta q(\nu t),$$

which has the **4** parameters: α, β, ω, and ν.

We can reduce the parameters to **2** by rescaling the dependent variable, y, and time, t:

$$y = Az \qquad \text{and} \qquad \tau = Bt.$$

Positive: Easier to determine types of ***qualitative behavior*** and fewer parameters to fit.

Negative: Scaled parameters may not match natural kinetic parameters and may be hard to unravel fitting experiments.

Scaling Parameters in ODE

The scaled **ODE** becomes:

$$AB\frac{dz}{d\tau} + A\alpha p\left(\tfrac{\omega}{B}\tau\right)z = \beta q\left(\tfrac{\nu}{B}\tau\right),$$

which is equivalent to

$$\frac{dz}{d\tau} + \tfrac{\alpha}{B}p\left(\tfrac{\omega}{B}\tau\right)z = \tfrac{\beta}{AB}q\left(\tfrac{\nu}{B}\tau\right).$$

This reduces to **2** parameters by taking:

$$\tfrac{\alpha}{B} = 1 \qquad \text{and} \qquad \tfrac{\beta}{AB} = 1,$$

or $B = \alpha$ and $A = \frac{\beta}{\alpha}$.

This becomes the ***scaled ODE***:

$$\frac{dz}{d\tau} + p\left(\hat{\omega}\tau\right)z = q\left(\hat{\nu}\tau\right),$$

where $\hat{\omega} = \omega/\alpha$ and $\hat{\nu} = \nu/\alpha$.

Perturbation 1

The previous scaled **ODE** gave:

$$\frac{dz}{d\tau} + p\left(\hat{\omega}\tau\right)z = q\left(\hat{\nu}\tau\right),$$

where $\hat{\omega} = \omega/\alpha$ and $\hat{\nu} = \nu/\alpha$.

Suppose the time arguments of p and q vary rapidly, so

$$\hat{\omega} = \tfrac{\omega}{\alpha} \gg 1 \qquad \text{and} \qquad \hat{\nu} = \tfrac{\nu}{\alpha} \gg 1,$$

or better written

$$\hat{\omega} = \tfrac{1}{\varepsilon}\hat{\omega}_0 \quad \text{and} \quad \hat{\nu} = \tfrac{1}{\varepsilon}\hat{\nu}_0 \qquad \text{with} \quad \varepsilon \gg 1 \quad \text{or} \quad \tfrac{1}{\varepsilon} \gg 1.$$

The **ODE** can be written:

$$\frac{dz}{d\tau} + p\left(\tfrac{\hat{\omega}_0}{\varepsilon}\tau\right)z = q\left(\tfrac{\hat{\nu}_0}{\varepsilon}\tau\right),$$

where p and q are varying quickly.

Perturbation 2

With the **ODE**:

$$\frac{dz}{d\tau} + p\left(\frac{\hat{\omega}_0}{\varepsilon}\tau\right) z = q\left(\frac{\hat{\nu}_0}{\varepsilon}\tau\right),$$

we scale again with $T = \tau/\varepsilon$, so $\frac{dz}{d\tau} = \frac{dz}{dT}\frac{dT}{d\tau} = \frac{1}{\varepsilon}\frac{dz}{dT}$.

This results in the following:

$$\frac{1}{\varepsilon}\frac{dz}{dT} + p\left(\hat{\omega}_0 T\right) z = q\left(\hat{\nu}_0 T\right),$$

or

$$\frac{dz}{dT} + \varepsilon p\left(\hat{\omega}_0 T\right) z = \varepsilon q\left(\hat{\nu}_0 T\right),$$

which is solved from our ***linear ODE*** technique.

Perturbation 3

The ***integrating factor*** is:

$$\mu(T) = e^{\varepsilon \int_0^T p(\hat{\omega}_0 s)ds},$$

which gives the solution:

$$z(T) = z(0)e^{-\varepsilon \int_0^T p(\hat{\omega}_0 s)ds} + \varepsilon e^{-\varepsilon \int_0^T p(\hat{\omega}_0 s)ds} \int_0^T q(\hat{\nu}_0 s) e^{\varepsilon \int_0^s p(\hat{\omega}_0 u)du} ds.$$

However, by **Taylor's Theorem**:

$$e^{-\varepsilon \int_0^T p(\hat{\omega}_0 s)ds} = 1 - \varepsilon \int_0^T p(\hat{\omega}_0 s)ds + \frac{\varepsilon^2}{2!}\left(\int_0^T p(\hat{\omega}_0 s)ds\right)^2 - \dots$$

Perturbation 4

With this expansion of $e^{-\varepsilon\int_0^T p(\hat{\omega}_0 s)ds}$ inserted into the solution:

$$z(T) = z(0)e^{-\varepsilon\int_0^T p(\hat{\omega}_0 s)ds} + \varepsilon e^{-\varepsilon\int_0^T p(\hat{\omega}_0 s)ds}\int_0^T q(\hat{\nu}_0 s)e^{\varepsilon\int_0^s p(\hat{\omega}_0 u)du}ds.$$

we can collect the terms based on the order of ε and obtain:

$$z(T) = z(0) + \varepsilon\left(\int_0^T q(\hat{\nu}_0 s)ds - z(0)\int_0^T p(\hat{\omega}_0 s)ds\right) + \mathcal{O}\left(\varepsilon^2\right).$$

For ε small the $\mathcal{O}\left(\varepsilon^2\right)$ terms are insignificant, which means the solution is approximated by the first two terms of the ε expansion, a significant reduction in computation.

Perturbation Example 1

Example: Consider the following ***linear ODE***:

$$\frac{dy}{dt} + \alpha\sin(\omega t)y = \beta\cos(\omega t).$$

From before we scale the problem with

$$y = Az \quad \text{and} \quad \tau = Bt, \qquad \text{taking} \qquad B = \alpha \quad \text{and} \quad A = \frac{\beta}{\alpha}.$$

The rescaled problem is

$$\frac{dz}{d\tau} + \sin\left(\frac{\omega}{\alpha}\tau\right)z = \cos\left(\frac{\omega}{\alpha}\tau\right).$$

Perturbation Example 2

Example: Suppose that $\omega = 10$ and $\alpha = 0.1$, so $\frac{\omega}{\alpha} \equiv \frac{1}{\varepsilon} = 100 \gg 1$.

The rescaled problem becomes

$$\frac{dz}{d\tau} + \sin\left(\frac{\tau}{\varepsilon}\right) z = \cos\left(\frac{\tau}{\varepsilon}\right).$$

With $T = \frac{\tau}{\varepsilon}$, the ***linear ODE*** is

$$\frac{dz}{dT} + \varepsilon \sin(T)\, z = \varepsilon \cos(T),$$

which has the solution:

$$z(T) = z(0)e^{\varepsilon(\cos(T)-1)} + \varepsilon e^{\varepsilon(\cos(T)-1)} \int_0^T \cos(s) e^{-\varepsilon(\cos(s)-1)} ds.$$

Perturbation Example

From **Taylor's Theorem**, we have

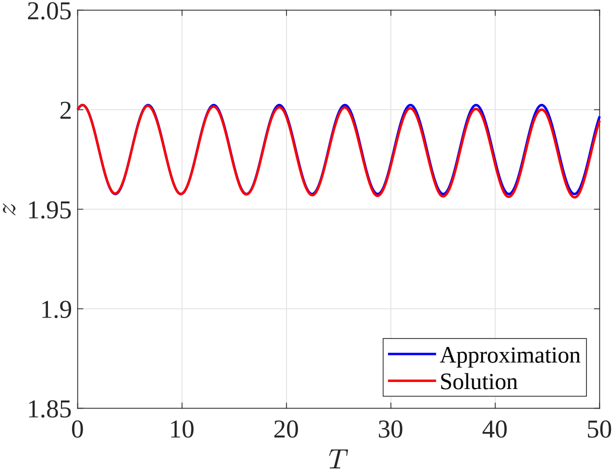
$$e^{\varepsilon(\cos(T)-1)} = 1 + \varepsilon(\cos(T) - 1) + \mathcal{O}\left(\varepsilon^2\right).$$

It follows that the solution can be approximated by

$$\begin{aligned} z(T) &= z(0)\left(1 + \varepsilon(\cos(T) - 1) + \mathcal{O}\left(\varepsilon^2\right)\right) \\ &\quad + \varepsilon\left(1 + \varepsilon(\cos(T) - 1) + \mathcal{O}\left(\varepsilon^2\right)\right) \int_0^T \cos(s)\left(1 - \varepsilon(\cos(s) - 1) + \mathcal{O}\left(\varepsilon^2\right)\right) ds, \\ z(T) &= z(0) + \varepsilon\left(z(0)(\cos(T) - 1) + \int_0^T \cos(s) ds\right) + \mathcal{O}\left(\varepsilon^2\right), \\ z(T) &= z(0) + \varepsilon\Big(z(0)(\cos(T) - 1) + \sin(T)\Big) + \mathcal{O}\left(\varepsilon^2\right). \end{aligned}$$

Perturbation Example

The figure below shows both the approximate solution and actual solution for $z(0) = 2$.



Perturbation Example

The figure was produced by the **MatLab** program below.

```
z0 = 2; ep = 0.01;
tt = linspace(0,50,500);
zz = z0 + ep*(z0*(cos(tt)-1) + sin(tt));
[t1,z1] = ode23(@perturb,tt,z0);
plot(tt,zz,'b-','LineWidth',1.5);
hold on
plot(t1,z1,'r-','LineWidth',1.5);
grid;
legend('Approximation', 'Solution','Location','southeast');
xlim([0,50]);
ylim([1.85,2.05]);
```

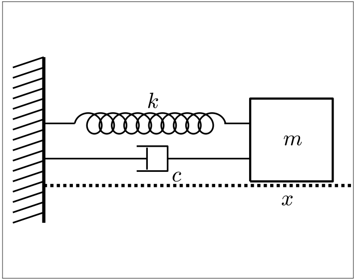
```
function zp = perturb(T,z)
%Perturbation ODE
ep = 0.01;
zp = -ep*sin(T)*z + ep*cos(T);
end
```

Mass-Spring 1

Mass-Spring Example: Consider a mass-spring system with spring constant, k, and damping proportional to the velocity of the mass, $c\dot{x}$.

Newton's Law gives:

$$m\ddot{x} = -kx - c\dot{x}.$$



This gives the **ODE**:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0,$$

where $c \geq 0$ and $m, k > 0$.

We ***scale*** the time with $\tau = \beta t$, so

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \beta\frac{d}{d\tau}, \qquad \text{and}$$

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \frac{d}{dt}\left(\beta\frac{d}{d\tau}\right) = \beta\frac{d}{dt}\left(\frac{d}{d\tau}\right) = \beta\frac{d^2}{d\tau^2}\left(\frac{d\tau}{dt}\right) = \beta^2\frac{d^2}{d\tau^2}.$$

Mass-Spring

Mass-Spring Example: Let $x(t) = \hat{x}(\beta t) = \hat{x}(\tau)$, then this scaling gives:

$$\beta^2 \frac{d^2\hat{x}}{d\tau^2} + \frac{c\beta}{m}\frac{d\hat{x}}{d\tau} + \frac{k}{m}\hat{x} = 0.$$

Let $\beta^2 = \frac{k}{m}$ and define $\gamma = \frac{c}{\sqrt{mk}}$, then ignoring the hats, the ***scaled mass-spring system with damping*** is:

$$\ddot{x} + \gamma\dot{x} + x = 0.$$

This 2^{nd} **order ODE** is transformed into a system of 1^{st} **order ODEs** by letting $x_1(\tau) = x(\tau)$ and $x_2(\tau) = \dot{x}(\tau)$, so

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - \gamma x_2, \end{aligned}$$

which in matrix form is:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{or} \qquad \dot{\mathbf{x}} = A\mathbf{x}.$$

Mass-Spring System 1

Mass-Spring System: The matrix given by:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix},$$

satisfies the ***characteristic equation***:

$$\det|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda - \gamma \end{vmatrix} = \lambda^2 + \gamma\lambda + 1 = 0.$$

The ***eigenvalues*** satisfy:

$$\lambda = \frac{1}{2}\left[-\gamma \pm \sqrt{\gamma^2 - 4}\right].$$

There are **3 cases**, which we examine.

Mass-Spring System 2

Case (i): Consider $\gamma^2 - 4 > 0$ (or $c > 2\sqrt{mk}$), which leads to **two real negative eigenvalues**,

$$\lambda_1 < \lambda_2 < 0.$$

The ***associated eigenvectors*** solve:

$$(A - \lambda_i I)\mathbf{v}_i = \begin{pmatrix} -\lambda_i & 1 \\ -1 & -\lambda_i - \gamma \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives the eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}.$$

The solution of the ***mass-spring*** problem is:

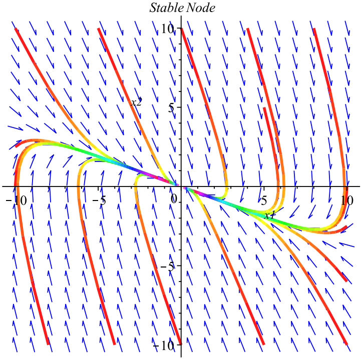
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} e^{\lambda_2 t}.$$

Mass-Spring System

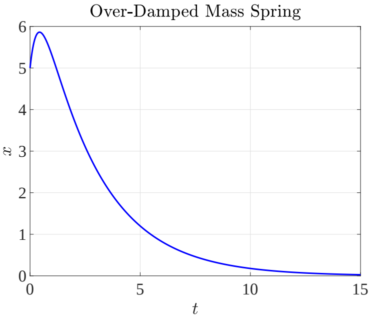
Case (i): Consider the specific case when $\gamma = 3$, so

$$\lambda_1 = 0.5(-3 - \sqrt{5}) \approx -2.618 \quad \text{and} \quad \lambda_2 = 0.5(-3 + \sqrt{5}) \approx -0.382.$$

This is the ***over-damped mass-spring*** problem. Below is the solution $x(0) = 5$ and $\dot{x}(0) = 5$.



Phase Portrait



Time Series

Mass-Spring System

Case (ii): Consider $\gamma^2 - 4 < 0$ (or $c < 2\sqrt{mk}$), which leads to **two complex eigenvalues** with **negative real part** (unless $\gamma = 0$),

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2} = -\frac{\gamma}{2} \pm i\omega, \qquad \omega = \frac{\sqrt{4-\gamma^2}}{2}, \quad (0 < \gamma < 2).$$

The ***associated eigenvectors*** solve:

$$(A - \lambda_1 I)\mathbf{v}_1 = \begin{pmatrix} -\lambda_1 & 1 \\ -1 & -\lambda_1 - \gamma \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives the eigenvector:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{\gamma}{2} + i\omega \end{pmatrix}.$$

The solution of the ***mass-spring*** problem is:

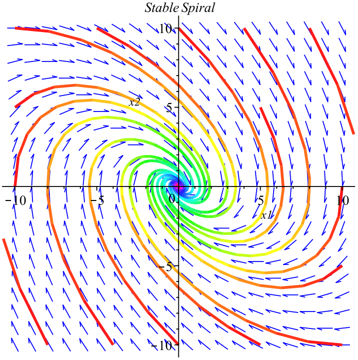
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{-\gamma t/2}\left[c_1 \begin{pmatrix} \cos(\omega t) \\ -\frac{\gamma}{2}\cos(\omega t) - \omega\sin(\omega t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(\omega t) \\ \omega\cos(\omega t) - \frac{\gamma}{2}\sin(\omega t) \end{pmatrix}\right].$$

Mass-Spring System

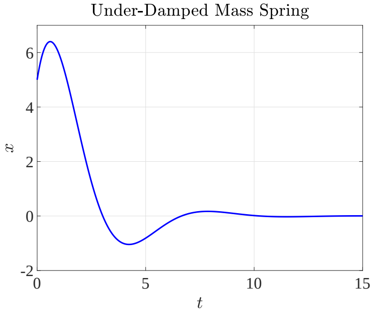
Case (ii): Consider the specific case when $\gamma = 1$, so

$$\lambda_{1,2} = 0.5(-1 \pm i\sqrt{3}) \approx -0.5 \pm 0.866i.$$

This is the ***under-damped mass-spring*** problem. Below is the solution $x(0) = 5$ and $\dot{x}(0) = 5$.



Phase Portrait



Time Series

Mass-Spring System

When $\gamma = 0$, this system is ***undamped*** and the resulting solution produces a center as shown in the **Introduction** notes.

Case (iii): Consider $\gamma^2 - 4 = 0$ (or $c = 2\sqrt{mk}$), which leads to the **repeated eigenvalue**, $\lambda = -1$.

The ***associated eigenvector*** solves:

$$(A+I)\mathbf{v}_1 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \text{so} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This eigenvalue has an ***algebraic multiplicity*** $= 2$ and a ***geometric multiplicity*** $= 1$.

The second solution to the **ODE** comes from the ***higher null space***. (Solve $(A+I)\mathbf{w} = \mathbf{v}$.)

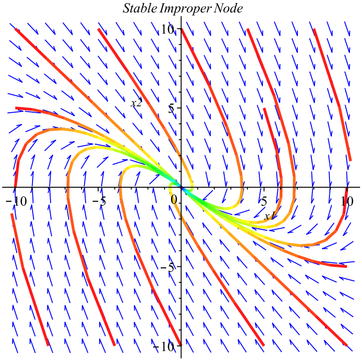
The solution of this ***mass-spring*** problem is:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{-t}.$$

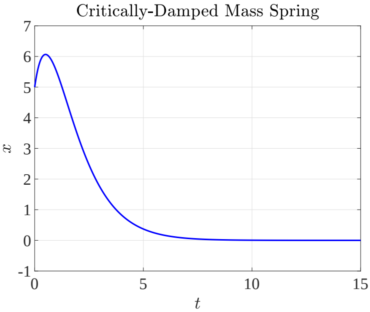
Later we'll investigate details of this case more.

Mass-Spring System

Case (iii): This is the ***critically-damped mass-spring*** problem. Below is the solution $x(0) = 5$ and $\dot{x}(0) = 5$.



Phase Portrait



Time Series