Math 537 - Ordinary Differential Equations

Lecture Notes – Linear Systems and Fundamental Solution

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Outline

1. Linear Systems of ODEs
   - Definitions and Matrix Properties
   - Matrix Diagonalization
   - Jordan Canonical Form
2. Fundamental Solution
   - Jordan Form and Complex Eigenvalues
   - Stability of 2×2 Systems
3. General Linear System
   - Homogeneous System
   - Linear Nonhomogeneous System

Example 1

Example 1: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -0.5 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

Since this is a diagonal matrix, we obtain the ***eigenvalues*** from the diagonal elements, $\lambda_1 = -0.5$ and $\lambda_2 = -1$.

The ***characteristic equation*** is

$$\det \begin{vmatrix} -0.5 - \lambda & 0 \\ 0 & -1 - \lambda \end{vmatrix} = (\lambda + 0.5)(\lambda + 1) = 0.$$

For $\lambda_1 = -0.5$, we have the ***associated eigenvector*** $\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Similarly, for $\lambda_2 = -1$ we have the ***associated eigenvector*** $\xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Example 2

Example 1 (cont): The general solution satisfies:

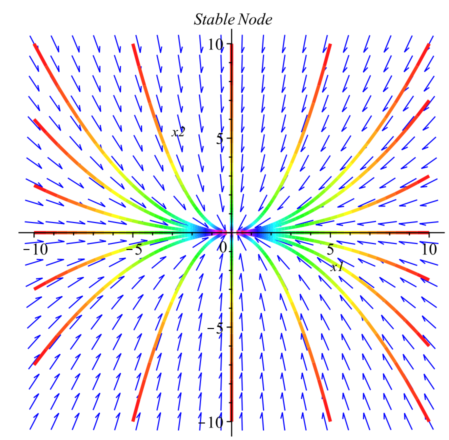
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-0.5t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t},$$

which is a solution exponentially decaying toward the origin.

This is a **sink** or **stable node**.

Solutions move more rapidly in the direction $\xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, while decaying more slowly in the direction $\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

This example shows how easy it is to solve systems of differential equations with diagonal matrices, since the variables are *uncoupled*.



Example 3

Example 1 (cont): The general solution is given by:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-0.5t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t},$$

so the linearly independent solutions are combined to give a ***fundamental solution***:

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^{-0.5t} & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

It is readily seen that

$$\dot{\mathbf{\Phi}} = A\mathbf{\Phi}, \qquad \text{and} \qquad \mathbf{\Phi}(0) = I.$$

Furthermore, any solution can be written:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \mathbf{\Phi}(t)\tilde{\mathbf{c}},$$

where $\tilde{\mathbf{c}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

Norms

We consider vectors $x \in \mathbb{R}^n$ (or $\mathbb{C}^n$) and define a "distance" in terms of the norm of a vector.

Definition (l_p Norm)

Consider an n-dimensional vector $x = [x_1, ..., x_n]^T \in \mathbb{R}^n$ (or $\mathbb{C}^n$). The **l_p norm** for the vector x is defined by the following:

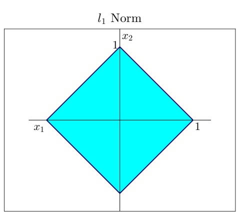
$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

Almost always the norms use $p = 1$ (taxicab or grid), $p = 2$ (Euclidean or distance), or $p = \infty$ (max)

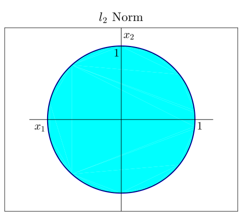
For $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we have $\|x\|_2 = (x_1^2 + x_2^2))^{1/2}$

Unit Circles

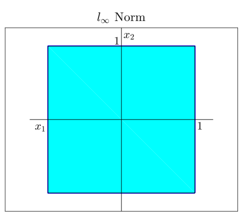
Consider $x \in \mathbb{R}^2$ and $\|x\| \le 1$ in **three** different **norms**



$\|x\|_1 \le 1$
or
$|x_1| + |x_2| \le 1$



$\|x\|_2 \le 1$
or
$\left(|x_1|^2 + |x_2|^2\right)^{1/2} \le 1$



$\|x\|_\infty \le 1$
or
$\max\{|x_1|, |x_2|\} \le 1$

Norms

Let $x = [x_1, ..., x_n]^T \in \mathbb{R}^n$, then the **norms** for $p = 1$, $p = 2$, or $p = \infty$ satisfy:

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^{n} |x_i| \\ \|x\|_2 &= \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \\ \|x\|_\infty &= \max_i \{|x_i|\} \end{aligned}$$

Property (Norm)

Given an n-dimensional vector $x = [x_1, ..., x_n]^T$, then:

$$\begin{aligned} \|x\| > 0, \qquad & \textit{if } x_i \ne 0 \textit{ for some } i, \\ \|x\| = 0, \qquad & \textit{if } x_i = 0 \textit{ for all } i. \end{aligned}$$

Norm – Example

Example: Consider $x = [0.2, 0.4, 0.6, 0.8]$.

- For $p = 1$,

$$\|x\|_1 = \sum_{i=1}^{4} |x_i| = 0.2 + 0.4 + 0.6 + 0.8 = 2.0$$

- **MatLab** command is `norm(x,1)`
- For $p = 2$,

$$\|x\|_2 = \left(\sum_{i=1}^{4} |x_i|^2 \right)^{1/2} = \sqrt{0.04 + 0.16 + 0.36 + 0.64} = 1.0954$$

- **MatLab** command is `norm(x)` or `norm(x,2)`
- For $p = \infty$,

$$\|x\|_\infty = \max_i |x_i| = 0.8$$

- **MatLab** command is `norm(x,inf)`

Cauchy-Schwarz Inequality and Equivalence

Property (Cauchy-Schwarz Inequality)

Consider two vectors, $\mathbf{x} = [x_1, \dots, x_n]^T$ and $\mathbf{y} = [y_1, \dots, y_n]^T$, in $\mathbb{R}^n$ (or $\mathbb{C}^n$). Then

$$\sum_{j=1}^{n} |x_j||y_j| \le \left(\sum_{j=1}^{n} |x_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} |y_j|^2 \right)^{\frac{1}{2}}.$$

Definition (Norm Equivalency)

Two norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are said to be ***equivalent*** if there exist constants C and D and $\mathbf{x} \in \mathbb{R}^n$ (or $\mathbb{C}^n$) such that

$$C\|\mathbf{x}\|_\alpha \le \|\mathbf{x}\|_\beta \le D\|\mathbf{x}\|_\alpha.$$

If norms are equivalent, then it doesn't really matter which norm is used for showing different properties.

Norm Equivalence 1

It is easy to see with the ***Cauchy-Schwarz inequality*** that

$$\begin{aligned} \|\mathbf{x}\|_1 = \sum_{j=1}^{n} |x_j| = \sum_{j=1}^{n} |x_j| \cdot 1 &\le \left(\sum_{j=1}^{n} |x_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} 1 \right)^{\frac{1}{2}} \\ &= \sqrt{n}\|\mathbf{x}\|_2 \end{aligned}$$

If $\|\mathbf{x}\|_1 = K$, then $|x_j| \le K$, so

$$\begin{aligned} \|\mathbf{x}\|_2 = \left(\sum_{j=1}^{n} |x_j|^2 \right)^{\frac{1}{2}} &\le \left(\sum_{j=1}^{n} K|x_j| \right)^{\frac{1}{2}} \\ &\le \sqrt{K}\|\mathbf{x}\|_1^{\frac{1}{2}} = K = \|\mathbf{x}\|_1. \end{aligned}$$

It follows that $\|\cdot\|_1$ and $\|\cdot\|_2$ are ***equivalent*** as

$$\frac{1}{\sqrt{n}}\|\mathbf{x}\|_1 \le \|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1.$$

Norm Equivalence 2

Relating to $\|\cdot\|_\infty$, we see immediately that

$$\|\mathbf{x}\|_1 = \sum_{j=1}^{n} |x_j| \le \sum_{j=1}^{n} \|\mathbf{x}\|_\infty = n\|\mathbf{x}\|_\infty,$$

and clearly $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_1$, so

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_1 \le n\|\mathbf{x}\|_\infty,$$

which gives ***equivalency*** of the $\|\cdot\|_1$ and $\|\cdot\|_\infty$ norms.

All of this can be strung together to show that:

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1 \le \sqrt{n}\|\mathbf{x}\|_2 \le n\|\mathbf{x}\|_\infty,$$

which means that all of these ***norms*** are ***equivalent***.

Norm Equivalence 3

The fact that all these **norms are equivalent** means that one can use whatever norm is most convenient.

The bounds will change, but we obtain limits on our estimates.

Depending on what we are attempting to accomplish, we will choose different norms, each with their own special properties.

The $\|\cdot\|_2$ is particularly important as

$$\|\mathbf{x}\|_2 = (\langle \mathbf{x}, \mathbf{x} \rangle)^{\frac{1}{2}},$$

where

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{n} x_j y_j^*$$

is an ***inner-product***, providing important structure to our space.

$\|\cdot\|_1$ and $\|\cdot\|_\infty$ do **NOT** come from ***inner-products***.

Norm of a Matrix

Consider matrices $A : \mathbb{C}^n \to \mathbb{C}^n$ and $B : \mathbb{C}^n \to \mathbb{C}^n$.

Property (Matrix Norm)

*A **matrix norm** on the set of all $n \times n$ matrices is a real-valued function, $\|\cdot\|$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:*

1. $\|A\| \geq 0$ *(positivity);*
2. $\|A\| = 0$, *if and only if A is $\mathbf{0}$, the matrix with all entries 0;*
3. $\|\alpha A\| = |\alpha|\|A\|$ *(scalar multiplication);*
4. $\|A + B\| \leq \|A\| + \|B\|$ *(triangle inequality);*
5. $\|AB\| \leq \|A\|\|B\|$ *(sub-multiplicative norm);*

p-Norm of a Matrix

p-Norm of a Matrix: There are a number of norms on a matrix. The most common norm for a matrix is defined by the vector p-norms for $\mathbb{R}^n$

Definition (Matrix p-Norm)

If $\|\cdot\|_p$ is a vector norm on $\mathbb{R}^n$, then

$$\|A\|_p = \max_{\|x\|_p = 1} \|Ax\|_p = \max_{\|x\|_p \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

is a matrix norm.

The Matrix p-norm gives the relative expansion of matrix A

It follows that for any x

$$\|A\|_p \geq \frac{\|Ax\|_p}{\|x\|_p} \qquad \text{or} \qquad \|Ax\|_p \leq \|A\|_p \|x\|_p$$

p-Norm of a Matrix – Special Cases

When A is applied to a unit vector $\|x\|_p$, then $\|A\|_p$ is the largest image of $\|Ax\|_p$ from all $\|x\|_p = 1$

Our primary interests are the cases $p = 1, 2, \infty$, which are readily computable

- $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| =$ maximum absolute column sum
- $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| =$ maximum absolute row sum
- $\|A\|_2 = \sqrt{\lambda_{max}(A^*A)} = \sigma_{max}(A)$, which is the square root of the largest eigenvalue of A^*A, where A^* is the conjugate transpose of A. $\sigma_{max}(A)$ is the largest singular value of A

Example

Example: Consider
$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Computing the **2 norm**:

$$\|A\mathbf{x}\|_2 = \left(|\lambda_1|^2|x_1|^2 + |\lambda_2|^2|x_2|^2\right)^{\frac{1}{2}}.$$

If $|\lambda_1| > |\lambda_2|$, then choose $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and it follows that

$$\left\| A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_2 = |\lambda_1|,$$

so $\|A\|_2 = |\lambda_1|$.

Similarity and Exponential of Matrix

There are a number of definitions about matrices that are needed.

Definition (Similar Matrices)

Consider two $n \times n$ matrices, A, B. Matrix A is ***similar*** to B if there exists an invertible matrix P such that
$$AP = PB \qquad \text{or} \qquad B = P^{-1}AP.$$

Fact: ***Similar matrices*** have the same ***characteristic equation***.

The exponential of a matrix is defined by a ***Taylor's series***.

Definition (e^A)

Let A be an $n \times n$ matrix. The ***matrix exponential*** is defined by the following series:
$$e^A = I + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Exponential of Matrix

The ***exponential of matrix*** is defined by the sum of the series:
$$e^A = I + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

This series only makes sense if it converges.

We show this series converges for any matrix $A : \mathbb{C}^n \to \mathbb{C}^n$ by defining the ***partial sums*** and applying the ***Cauchy criterion*** for sequences.

$$S_k = I + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!}.$$

From the ***sub-multiplicative norm property***, $\|A^n\| \le \|A\|^n$.

The partial sums give for $m > p$

$$\|S_m - S_p\| = \left\| \sum_{k=p+1}^{m} \frac{A^k}{k!} \right\| \le \sum_{k=p+1}^{m} \frac{\|A^k\|}{k!} \le \sum_{k=p+1}^{m} \frac{\|A\|^k}{k!}.$$

Since $\|A\|$ is a real number, from Calculus we know this last quantity can be made arbitrarily small for sufficiently large p; and thus, this converges by the ***Cauchy criterion***.

e^{At} Properties and Example

Property (Matrix Exponential Product)

If M and P commute ($MP = PM$), then
$$e^M \cdot e^P = e^{M+P}.$$

Example: Find e^{At}, where $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Since the last two matrices commute, we have

$$\begin{aligned} e^{At} &= exp\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} t \cdot exp\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t \\ &= \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{pmatrix} \left[I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 \frac{t^2}{2!} + \dots \right]. \end{aligned}$$

However, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so the infinite series terminates after **2** terms. Thus,
$$e^{At} = e^{3t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{3t} & te^{3t} \\ 0 & e^{3t} \end{pmatrix}.$$

Diagonalization 1

Consider the system of **ODEs** with $A(n\times n)$

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where A has n distinct real eigenvalues.

From Linear Algebra we have the following Theorem:

Theorem (Diagonalization)

Assume the matrix $A(n\times n)$ has the real distinct eigenvalues, $\lambda_1,\lambda_2,\dots\lambda_n$, then any set of corresponding eigenvectors, $\{\mathbf{v}_1,\mathbf{v}_2,\dots\mathbf{v}_n\}$ forms a basis of $\mathbb{R}^n$, the matrix $P=[\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_n]$ is invertible, and

$$P^{-1}AP = D = diag[\lambda_1,\lambda_2,\dots\lambda_n].$$

Proof: Using the definition of eigenvalues and properties of matrices,

$$\begin{aligned} P^{-1}AP &= P^{-1}A[\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_n] = P^{-1}[A\mathbf{v}_1,A\mathbf{v}_2,\dots,A\mathbf{v}_n] \\ &= P^{-1}[\lambda_1\mathbf{v}_1,\lambda_2\mathbf{v}_2,\dots,\lambda_n\mathbf{v}_n] \\ &= [\lambda_1P^{-1}\mathbf{v}_1,\lambda_2P^{-1}\mathbf{v}_2,\dots,\lambda_nP^{-1}\mathbf{v}_n]. \end{aligned}$$

Diagonalization 2

Proof (cont.): However, $\mathbf{v}_j$ is the j^{th} column of P and

$$P^{-1}\mathbf{v}_j = j^{th} \text{ column of } P^{-1}P = j^{th} \text{ column of } I,$$

which implies $P^{-1}AP = D$. *q.e.d.*

Returning to our **ODE** with $\dot{\mathbf{x}} = A\mathbf{x}$, we define the ***linear transformation***

$$\mathbf{y} = P^{-1}\mathbf{x},$$

where P is defined in the Theorem above.

It follows that

$$\begin{aligned} \mathbf{x} &= P\mathbf{y}, \\ \dot{\mathbf{y}} = P^{-1}\dot{\mathbf{x}} &= P^{-1}A\mathbf{x} = P^{-1}AP\mathbf{y}, \end{aligned}$$

which leaves the ***uncoupled linear system***:

$$\dot{\mathbf{y}} = D\mathbf{y} = diag[\lambda_1,\lambda_2,\dots\lambda_n]\mathbf{y}.$$

Diagonalization 3

The ***uncoupled linear system***:

$$\dot{\mathbf{y}} = D\mathbf{y} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}\mathbf{y}$$

has the solution:

$$\mathbf{y}(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{\lambda_n t} \end{pmatrix}\mathbf{y}(0) \equiv e^{Dt}\mathbf{y}(0).$$

With $\mathbf{y}(0) = P^{-1}\mathbf{x}(0)$ and $\mathbf{x}(t) = P\mathbf{y}(t)$ the solution to the original problem becomes:

$$\mathbf{x}(t) = P\begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{\lambda_n t} \end{pmatrix}P^{-1}\mathbf{x}(0) \equiv e^{At}\mathbf{x}(0).$$

Example 1 1

Example 1: Consider the following system of **ODEs**:

$$\dot{\mathbf{x}} = \begin{pmatrix} 3 & 0 & -4 \\ -4 & 2 & 7 \\ 2 & 0 & -3 \end{pmatrix}\mathbf{x}.$$

With the help of **Maple**, we find the ***eigenvalues*** and ***associated eigenvectors***:

$$\lambda_1 = 2, \quad \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \lambda_2 = 1, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \qquad \lambda_3 = -1, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

It follows that we want the following ***transformation matrix***:

$$P = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \text{with} \quad P^{-1} = \begin{pmatrix} -2 & 1 & 3 \\ 1 & 0 & -1 \\ -1 & 0 & 2 \end{pmatrix}$$

where again **Maple** helps us with the ***inverse matrix***.

Example 1 2

Example 1: From our Theorem we have:

$$P^{-1}AP = D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

With the ***linear transformation*** $\mathbf{y} = P^{-1}\mathbf{x}$, we obtain the ***uncoupled system***:

$$\dot{\mathbf{y}} = D\mathbf{y},$$

which has the solution:

$$\mathbf{y}(t) = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \mathbf{y}(0).$$

Transforming the system back to the original coordinates gives:

$$\mathbf{x}(t) = P \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} P^{-1}\mathbf{x}(0) = \begin{pmatrix} 2\,e^{t} - e^{-t} & 0 & -2\,e^{t} + 2\,e^{-t} \\ -2\,e^{2\,t} + e^{t} + e^{-t} & e^{2\,t} & 3\,e^{2\,t} - e^{t} - 2\,e^{-t} \\ e^{t} - e^{-t} & 0 & -e^{t} + 2\,e^{-t} \end{pmatrix} \mathbf{x}(0).$$

Example 1 3

Example 1: From above, our solution in the transformed coordinates satisfies:

$$\mathbf{y}(t) = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \mathbf{y}(0).$$

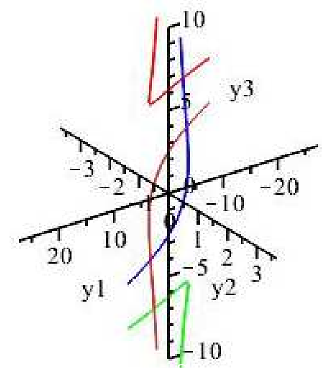
Below we see a graph showing several trajectories for this solution.

The **4** trajectories begin near the y_3-axis, then asymptotically approach the y_1y_2-plane.

This system has an **Unstable Node** in the y_1 vs y_2 plane ($y_3 = 0$).

This system has **Saddle Nodes** in the y_1 vs y_3 plane ($y_2 = 0$) or y_2 vs y_3 plane ($y_1 = 0$).

Behavior is best viewed in the 2D projections. See **Maple** worksheet.



Jordan Canonical Form 1

When the system of **ODEs** with $A(n \times n)$

$$\dot{\mathbf{x}} = A\mathbf{x},$$

has the ***algebraic multiplicities*** of eigenvalues of A agree with the ***geometric multiplicities***, then we can ***diagonalize*** the matrix with the n linearly independent ***eigenvectors*** and readily solve the ***uncoupled system***.

However, there are times when the ***geometric multiplicities*** are less than the ***algebraic multiplicities***, and the matrix A cannot be ***diagonalized***.

Definition (Generalized Eigenspace)

Let $A : V \to V$ be a linear transformation on a complex vector space, and let λ be a complex number. The ***generalized λ-eigenspace***, W_λ, is the subspace of V consisting of vectors $\mathbf{v} \in V$ such that

$$(A - \lambda I)^m \mathbf{v} = \mathbf{0},$$

for some positive integer m. The vector $\mathbf{v}$ is said to be a ***generalized eigenvector of rank m***, if m is the smallest positive integer such that $\mathbf{v}$ is in the kernel of $(A - \lambda I)^m$.

Jordan Canonical Form 2

Theorem (Jordan Canonical Form)

For each complex constant $n \times n$ matrix A, there exists a nonsingular matrix P such that the matrix $J = P^{-1}AP$ is in the canonical form:

$$J = \begin{pmatrix} J_0 & 0 & \dots & 0 \\ 0 & J_1 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_s \end{pmatrix},$$

where J_0 is a diagonal matrix with diagonal elements, $\lambda_1, \lambda_2, \dots, \lambda_k$, (not necessarily distinct) and each J_p is an $n_p \times n_p$ matrix of the forms:

$$J_0 = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_k \end{pmatrix} \quad \text{and} \quad J_p = \begin{pmatrix} \lambda_{k+p} & 1 & 0 & \dots & 0 \\ 0 & \lambda_{k+p} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & \lambda_{k+p} \end{pmatrix},$$

where $p = 1, \dots, s$ and λ_{k+p} need not differ from λ_{k+q} if $p \neq q$ and $k + n_1 + \dots + n_s = n$. The eigenvalues of A are $\lambda_i, i = 1, 2, \dots, k + s$ with the simple eigenvalues appearing in J_0.

Jordan Canonical Form: Maple

Maple provides a *toolbox (LinearAlgebra)* that easily computes the ***Jordan Canonical Form*** of a matrix.

A worksheet is available for the matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix}.$$

We show the commands **CharacteristicPolynomial(A,z)** and **Eigenvectors(A)**, giving the obvious results.

The command **JordanForm** allows finding the ***Jordan Canonical Form*** of A and the ***Transition Matrix***, Q, easily:

$$J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} \frac{1}{9} & \frac{2}{3} & \frac{8}{9} \\ \frac{2}{9} & -\frac{2}{3} & -\frac{2}{9} \\ \frac{4}{9} & \frac{2}{3} & -\frac{4}{9} \end{pmatrix}.$$

Fundamental Solution 1

Earlier we saw that if J_0 was a $k \times k$ diagonal matrix, then the solution of $\dot{\mathbf{x}} = J_0\mathbf{x}$ was

$$\mathbf{x}(t) = e^{J_0 t}\mathbf{x}(0),$$

where $e^{J_0 t} = diag[e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_k t}]$.

Next we evaluate $e^{J_p t}$, where $J_p = \lambda_{k+p} I_p + N_p$ and N_p is an $n_p \times n_p$ matrix:

$$N_p = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{pmatrix}.$$

It is easy to see that $\lambda_{k+p} I_p$ and N_p commute, so

$$e^{J_p t} = e^{\lambda_{k+p} t} \begin{pmatrix} 1 & t & \dots & \frac{t^{n_p-1}}{(n_p-1)!} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & t \\ 0 & \dots & \dots & 1 \end{pmatrix}.$$

Fundamental Solution 2

We saw that any matrix A can be transformed into ***Jordan canonical form***, J, which is in a block diagonal form with all the ***eigenvalues*** on the diagonal and repeated eigenvalues with an ***eigenspace*** having a ***kernel*** or ***nullspace*** larger than **1** having **ones** on the ***superdiagonal***.

The ***fundamental solution***, $\mathbf{\Psi}(t)$, of $\dot{\mathbf{y}} = J\mathbf{y}$ satisfies:

$$\mathbf{\Psi}(t) = e^{Jt} = \begin{pmatrix} e^{J_0 t} & 0 & \dots & 0 \\ 0 & e^{J_1 t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{J_s t} \end{pmatrix}.$$

because of the block structure of the matrix J.

It follows that the ***fundamental solution***, $\mathbf{\Phi}(t)$, of $\dot{\mathbf{x}} = A\mathbf{x}$ satisfies:

$$\mathbf{\Phi}(t) = e^{At} = e^{PJP^{-1}t} = Pe^{Jt}P^{-1}.$$

Example of Fundamental Solution 1

Example: Consider the system of linear homogeneous equations:

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{pmatrix} -7 & -5 & -3 \\ 2 & -2 & -3 \\ 0 & 1 & 0 \end{pmatrix}\mathbf{x}.$$

The ***characteristic equation*** satisfies:

$$\det \begin{pmatrix} -7-\lambda & -5 & -3 \\ 2 & -2-\lambda & -3 \\ 0 & 1 & -\lambda \end{pmatrix} = -(\lambda+3)^3 = 0,$$

implying A has the eigenvalue $\lambda = -3$ with ***algebraic multiplicity*** $= 3$.

Examining $A - \lambda I$ gives:

$$\begin{pmatrix} -7+3 & -5 & -3 \\ 2 & -2+3 & -3 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} -4 & -5 & -3 \\ 2 & 1 & -3 \\ 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix},$$

which is a ***rank*** **2** matrix, so $\ker(A+3I)$ is one-dimensional.

Example of Fundamental Solution 2

Example: Since $\ker(A+3I)$ is one-dimensional, the *geometric multiplicity* of $\lambda = -3$ is only **one**.

We compute $(A+3I)^2$ and $(A+3I)^3$ and find:

$$\begin{pmatrix} -4 & -5 & -3 \\ 2 & 1 & -3 \\ 0 & 1 & 3 \end{pmatrix}^2 = \begin{pmatrix} 6 & 12 & 18 \\ -6 & -12 & -18 \\ 2 & 4 & 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -4 & -5 & -3 \\ 2 & 1 & -3 \\ 0 & 1 & 3 \end{pmatrix}^3 = \mathbf{0},$$

which implies the *generalized eigenspace* has dimension **3**.

We create a *Jordan basis* by satisfying the following relations:

$$(A-\lambda I)\mathbf{v}_1 = \mathbf{0}, \qquad (A-\lambda I)\mathbf{v}_2 = \mathbf{v}_1, \qquad (A-\lambda I)\mathbf{v}_3 = \mathbf{v}_2.$$

The process employed is called a *Jordan chain*, where we select a vector $\mathbf{v}_3$ in the generalized eigenspace, which is $\mathbb{R}^3$ (which in this case cannot be in the eigenspace of $(A-\lambda I)^2$).

It suffices to take $\mathbf{v}_3 = [1,0,0]^T$.

Example of Fundamental Solution 3

Example: With $\mathbf{v}_3 = [1,0,0]^T$, we solve

$$\mathbf{v}_2 = (A-\lambda I)\mathbf{v}_3 = \begin{pmatrix} -4 & -5 & -3 \\ 2 & 1 & -3 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}$$

and

$$\mathbf{v}_1 = (A-\lambda I)\mathbf{v}_2 = \begin{pmatrix} -4 & -5 & -3 \\ 2 & 1 & -3 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ 2 \end{pmatrix}$$

Thus, we obtain our *linear transformation* matrix:

$$P = \begin{pmatrix} 6 & -4 & 1 \\ -6 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \quad \text{with} \quad P^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & 2 & 3 \end{pmatrix}.$$

It is not hard to see that

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} -7 & -5 & -3 \\ 2 & -2 & -3 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 6 & -4 & 1 \\ -6 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix} = J$$

Example of Fundamental Solution 4

Example: From our results before, the *fundamental solution* of $\dot{\mathbf{y}} = J\mathbf{y}$ is given by:

$$\boldsymbol{\Psi}(t) = e^{Jt} = e^{-3t}\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

The the *fundamental solution* of $\dot{\mathbf{x}} = A\mathbf{x}$ is given by:

$$\begin{aligned} \boldsymbol{\Phi}(t) &= e^{At} = Pe^{Jt}P^{-1} \\ &= \begin{bmatrix} 3e^{-3t}t^2 - 4e^{-3t}t + e^{-3t} & -5e^{-3t}t + 6e^{-3t}t^2 & -3e^{-3t}t + 9e^{-3t}t^2 \\ -3e^{-3t}t^2 + 2e^{-3t}t & e^{-3t}t + e^{-3t} - 6e^{-3t}t^2 & -3e^{-3t}t - 9e^{-3t}t^2 \\ e^{-3t}t^2 & e^{-3t}t + 2e^{-3t}t^2 & e^{-3t} + 3e^{-3t}t + 3e^{-3t}t^2 \end{bmatrix} \end{aligned}$$

The *general solution* of $\dot{\mathbf{x}} = A\mathbf{x}$ satisfies:

$$\mathbf{x}(t) = c_1e^{-3t}\mathbf{v}_1 + c_2e^{-3t}(t\mathbf{v}_1 + \mathbf{v}_2) + c_3e^{-3t}\left(\frac{t^2}{2!}\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3\right),$$

where $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are the respective columns of P.

Jordan Form and Complex Eigenvalues 1

What happens to the *Jordan canonical form* when some of the *eigenvalues* are *complex*?

If the *eigenvalues* come from a real matrix A and $\lambda_1 = \alpha - i\beta$, then $\lambda_2 = \alpha + i\beta$ is another eigenvalue.

Suppose that A is a 2×2 real matrix with *eigenvalues*, $\lambda = \alpha \pm i\beta$, then there exists a complex matrix P, such that

$$P^{-1}AP = J = \begin{pmatrix} \alpha - i\beta & 0 \\ 0 & \alpha + i\beta \end{pmatrix}.$$

Thus, a *fundamental solution (complex)* to $\dot{\mathbf{y}} = J\mathbf{y}$ satisfies:

$$\boldsymbol{\Psi}(t) = e^{Jt} = \begin{pmatrix} e^{(\alpha - i\beta)t} & 0 \\ 0 & e^{(\alpha + i\beta)t} \end{pmatrix}.$$

How are *real fundamental solutions* formed for this matrix A?

Jordan Form and Complex Eigenvalues 2

With the 2×2 real matrix A and $\lambda = \alpha \pm i\beta$, our theory gives the existence of a complex matrix P, such $P^{-1}AP = J$ is a ***diagonal matrix*** with the eigenvalues on the diagonal.

However, it is often preferable to transform A into the ***anti-symmetric matrix***, K:

$$K = Q^{-1}AQ = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where K is ***similar*** to A and Q has real entries.

Theorem (Complex Eigenvalues and Rotation-Scaling Matrices)

If the 2×2 real matrix A has eigenvalues $\alpha \pm i\beta$ (with $\beta \neq 0$), and if $\mathbf{v} + i\mathbf{w}$ is an eigenvector of A with eigenvalue $\alpha + i\beta$, then

$$Q^{-1}AQ = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = K, \qquad \text{where} \quad Q = [\mathbf{v}\ \mathbf{w}].$$

Jordan Form and Complex Eigenvalues 3

The previous theorem provides the tools for ***transforming*** the 2×2 real matrix A with a 2×2 real matrix Q into a ***similar*** 2×2 ***real anti-symmetric matrix***, K, which is a ***rotation-scaling matrix***.

This theorem generalizes to the higher dimensional ***eigenspaces*** to allow transformation of any real matrix A into a ***real Jordan form matrix***, where complex eigenvalues are represented by ***real anti-symmetric blocks*** on the diagonal.

It can be shown that the exponential of the ***anti-symmetric matrix***, K, has the following form:

$$e^{Kt} = e^{\alpha t} \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix},$$

which gives the ***fundamental solution*** to the **ODE**, $\dot{\mathbf{y}} = K\mathbf{y}$, given by

$$\mathbf{\Psi}(t) = e^{Kt}.$$

General Jordan Form with Complex Eigenvalues 1

Theorem (Real Jordan Canonical Form)

Let A be a real matrix with real eigenvalues, λ_j, $j = 1, \dots, k$ and complex eigenvalues, $\lambda_j = \alpha_j + \beta_j$ and $\bar{\lambda}_j = \alpha_j - \beta_j$, $j = k+1, \dots, n$. Then there exists a basis $\{v_1, \dots, v_k, u_{k+1}, w_{k+1}, \dots, u_n, w_n\}$ for $\mathbb{R}^{2n-k}$, where v_j, $j = 1, \dots, n$, are generalized eigenvectors of A, the first k of these are real and $u_j = \mathbf{Re}(v_j)$, $w_j = \mathbf{Im}(v_j)$ for $j = k+1, \dots, n$. The matrix $P = (v_1|\dots|v_k|u_{k+1}|w_{k+1}|\dots|u_n|w_n)$ is invertible with

$$P^{-1}AP = \begin{pmatrix} J_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & J_r \end{pmatrix},$$

*where the elementary Jordan blocks, J_i, $i = 1, \dots, r$ are either of the form of our previous Theorem for **Jordan Canonical Form** for the real eigenvalues, λ_j, $j = 1, \dots, k$, or of the form*

$$J_p = \begin{pmatrix} D_p & I_2 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & D_p & I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & D & I_2 \\ \mathbf{0} & \dots & \dots & \mathbf{0} & D_p \end{pmatrix},$$

where

$$D_p = \begin{pmatrix} \alpha_p & \beta_p \\ -\beta_p & \alpha_p \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

for $\lambda_p = \alpha_p + i\beta_p$ a complex eigenvalue of A.

General Jordan Form with Complex Eigenvalues 2

The ***Jordan Block matrices***, J_p, in the previous theorem coming from the complex eigenvalues, $\lambda_p, \bar{\lambda}_p$, depend on the ***algebraic*** and ***geometric multiplicities***.

For distinct complex eigenvalues or any complex pair, $\lambda_k = \alpha_k \pm i\beta_k$, with algebraic and geometric multiplicities agreeing have a diagonal form similar to J_0 in the previous theorem with diagonal elements,

$$D_k = \begin{pmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{pmatrix}.$$

When the complex pair, $\lambda_p = \alpha_p \pm i\beta_p$ has algebraic multiplicity $= 2m (m > 1)$ with geometric multiplicity $= 2$, then J_p has the form shown above with m diagonal blocks of the form D_p.

Fundamental Solution with Complex EVs 1

We use the theorem for the *real Jordan canonical form* to find the *Fundamental Solution* to the problem:

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0.$$

The Fundamental Solution satisfies:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = Pe^{Jt}P^{-1}\mathbf{x}_0.$$

We have seen the form of blocks of e^{Jt} for real eigenvalues and distinct complex eigenvalues.

Remains to show the block form of $e^{J_p t}$ for J_p from the theorem above with complex $\lambda_p = \alpha_p \pm i\beta_p$.

Fundamental Solution with Complex EVs 2

For the $2m \times 2m$ *Jordan Block matrix*, J_p, in the *real Jordan canonical form theorem*, it can be shown that the *Fundamental Solution*, $e^{J_p t}$, for $\lambda_p = \alpha_p \pm i\beta_p$ with algebraic multiplicity $= m$, has the form:

$$e^{J_p t} = e^{\alpha_p t}\begin{pmatrix} R & Rt & R\frac{t^2}{2!} & \dots & R\frac{t^{m-1}}{(m-1)!} \\ \mathbf{0} & R & Rt & \ddots & R\frac{t^{m-2}}{(m-2)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & R & Rt \\ \mathbf{0} & \dots & \dots & \mathbf{0} & R \end{pmatrix},$$

where R is the rotation matrix

$$R = \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}$$

and each entry in the solution block above being a 2×2 matrix.

Example with Complex EVs 1

Example: Consider the following system of linear homogeneous equations:

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -8 & -8 & -4 \end{pmatrix}\mathbf{x}.$$

The *characteristic equation* satisfies:

$$(\lambda^2 + 2\lambda + 2)^2 = 0,$$

which gives the *eigenvalues*, $\lambda = -1 \pm i$ with *algebraic multiplicity* of **2** each.

With the help of **Maple**, we obtain the *eigenvectors*:

$$\mathbf{v}_1 = (1, -1-i, 2i, 2-2i)^T \qquad \text{and} \qquad \mathbf{v}_2 = (1, -1+i, -2i, 2+2i)^T,$$

associated with $\lambda_1 = -1 - i$ and $\lambda_2 = -1 + i$, respectively.

However, these only have *geometric multiplicity* of **1** each.

Example with Complex EVs 2

Example: Maple readily gives the *Jordan canonical form* and its *transition matrix* for the complex solution:

$$J_c = \begin{pmatrix} -1-i & 1 & 0 & 0 \\ 0 & -1-i & 0 & 0 \\ 0 & 0 & -1+i & 1 \\ 0 & 0 & 0 & -1+i \end{pmatrix} \quad P_c = \begin{pmatrix} -\frac{1}{2}+\frac{i}{2} & \frac{1}{2}+i & -\frac{1}{2}-\frac{i}{2} & \frac{1}{2}-i \\ 1 & -i & 1 & i \\ -1-i & i & -1+i & -i \\ 2i & -2i & -2i & 2i \end{pmatrix},$$

with:

$$J_c = P_c^{-1}AP_c, \qquad \text{and} \qquad \mathbf{y} = P_c^{-1}\mathbf{x}.$$

This gives the *complex fundamental solution*:

$$\mathbf{y}(t) = e^{J_c t}\mathbf{y}(0) = \begin{pmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_2 t} & te^{\lambda_2 t} \\ 0 & 0 & 0 & e^{\lambda_2 t} \end{pmatrix}\mathbf{y}(0).$$

Thus, a *complex fundamental solution* to the $\dot{\mathbf{x}} = A\mathbf{x}$ satisfies:

$$\mathbf{\Phi}(t) = P_c e^{J_c t} P_c^{-1}.$$

Example with Complex EVs

Example: Our *real Jordan canonical form theorem* states we can find a matrix J *similar* to A in the following form:

$$J = \begin{pmatrix} -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix},$$

where $J = P^{-1}AP$ for some *transition matrix*, P.

This gives the *real fundamental solution*:

$$\Psi(t) = e^{Jt} = e^{-t}\begin{pmatrix} \cos(t) & \sin(t) & t\cos(t) & t\sin(t) \\ -\sin(t) & \cos(t) & -t\sin(t) & t\cos(t) \\ 0 & 0 & \cos(t) & \sin(t) \\ 0 & 0 & -\sin(t) & \cos(t) \end{pmatrix}.$$

Thus, a *real fundamental solution* to the $\dot{\mathbf{x}} = A\mathbf{x}$ satisfies:

$$\mathbf{\Phi}(t) = Pe^{Jt}P^{-1}.$$

Example with Complex EVs

Example: For the *fundamental solution* in $\mathbf{x}(t)$ the previous Slide shows that we need non-singular matrix P and P^{-1}, where A is *similar* to J.

A is a *companion matrix*, so *eigenvectors* have the form $\mathbf{v} = [1, \lambda, \lambda^2, \lambda^3]^T$.

The columns of P consists of the *eigenvectors* of A with the real and imaginary parts creating two columns of P for the real *Jordan canonical form*.

The second *eigenvector* comes from the second null space of A and takes more work to obtain the *transformation matrix*, P, for the *real Jordan canonical form* (see **Maple jordan** sheet):

$$P = \begin{pmatrix} 1 & 0 & 2 & 1 \\ -1 & 1 & -2 & 1 \\ 0 & -2 & 0 & -2 \\ 2 & 2 & 2 & 0 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 2 & 3 & \frac{5}{2} & 1 \\ -1 & -1 & -1 & 0 \\ -1 & -2 & -\frac{3}{2} & -\frac{1}{2} \\ 1 & 1 & \frac{1}{2} & 0 \end{pmatrix}$$

where $J = P^{-1}AP$.

Thus, a *real solution* to the $\dot{\mathbf{x}} = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$ is given by:

$$\mathbf{x}(t) = Pe^{Jt}P^{-1}\mathbf{x}_0.$$

Stability of 2×2 Systems

Consider the system

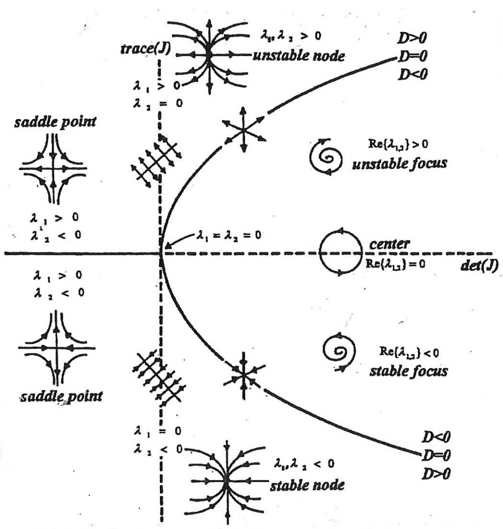
$$\dot{\mathbf{x}} = \mathbf{J}\mathbf{x},$$

where J is a 2×2 matrix.

Let λ_1 and λ_2 be eigenvalues of $\mathbf{J}\mathbf{x}$

Results from Linear Algebra give $tr(\mathbf{J}) = \lambda_1 + \lambda_2$, $\det|\mathbf{J}| = \lambda_1 \cdot \lambda_2$, and $D = (j_{11} - j_{22})^2 + 4j_{12}j_{21}$

The figure shows the **Stability Diagram** for $\dot{\mathbf{x}} = \mathbf{J}\mathbf{x}$ with axes of $tr(\mathbf{J})$ vs $\det|\mathbf{J}|$



General Linear System

Consider the general *linear system* given by:

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{g}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1}$$

where $A(t)$ is an $n \times n$ matrix and $\mathbf{g}(t)$ is an n vector.

Theorem (Existence and Uniqueness)

If $A(t)$ and $\mathbf{g}(t)$ are continuous on the interval $t \in [a, b]$ with $t_0 \in [a, b]$ and $\|\mathbf{x}_0\| < \infty$, then the system (1) has a unique solution, $\mathbf{\Phi}(t)$ satisfying the initial condition, $\mathbf{\Phi}(t_0) = \mathbf{x}_0$, and existing on the interval $t \in [a, b]$.

The proof of this theorem uses the continuity, hence boundedness of $A(t)$ and $\mathbf{g}(t)$ for $t \in [a, b]$. It also requires a *property* known as *Gronwall's inequality*. These details are left for the interested reader to explore.

General Homogeneous Linear System

Now consider the general ***linear homogeneous system*** given by:

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2}$$

where $A(t)$ is an $n \times n$ continuous matrix.

The previous theorem significantly states that there is the ***unique solution (trivial)*** $\mathbf{\Phi}_0(t) \equiv \mathbf{0}$, given the initial condition $\mathbf{x}_0 = \mathbf{0}$. (Inspection shows the ***trivial solution*** is always a solution to (2).)

Similarly, (2) has unique solutions $\mathbf{\Phi}_1(t), \mathbf{\Phi}_2(t), \dots, \mathbf{\Phi}_n(t)$ with $\mathbf{\Phi}_j(t_0) = \mathbf{e}_j$, where $\mathbf{e}_j$ is the j^{th} basis vector of $\mathbb{R}^n$.

The set $\{\mathbf{\Phi}_1(t), \mathbf{\Phi}_2(t), \dots, \mathbf{\Phi}_n(t)\}$ form a ***linearly independent set*** for $t \in [a, b]$.

General Homogeneous Linear System

Theorem (Solution Vector Space)

*If the complex $n \times n$ matrix $A(t)$ is continuous on an interval $t \in [a, b]$, then the solutions of the system (2) on $t \in [a, b]$ form a **vector space** of dimension n over the complex numbers.*

Let

$$\mathbf{\Phi}(t) = [\mathbf{\Phi}_1(t), \mathbf{\Phi}_2(t), \dots, \mathbf{\Phi}_n(t)]$$

be an $n \times n$ matrix created with the column solutions $\mathbf{\Phi}_j(t)$.

Clearly by the composition

$$\dot{\mathbf{\Phi}}(t) = A(t)\mathbf{\Phi}(t) \qquad \text{with} \quad \mathbf{\Phi}(t_0) = I.$$

The solution $\mathbf{\Phi}(t)$ forms a ***fundamental set of solutions*** to (2) on $t \in [a, b]$, where any solution:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c}$$

for some appropriate $\mathbf{c}$.

General Homogeneous Linear System

Theorem (Abel's Formula)

If $\mathbf{\Phi}(t)$ is a solution matrix of (2) on $t \in [a, b]$ and if $t_0 \in [a, b]$, then

$$\det \mathbf{\Phi}(t) = \det \mathbf{\Phi}(t_0) exp \left[\int_{t_0}^{t} \sum_{j=1}^{n} a_{jj}(s) ds \right], \qquad \text{for every } t \in [a, b].$$

It follows that either $\det \mathbf{\Phi}(t) \neq 0$ for each $t \in [a, b]$ or $\det \mathbf{\Phi}(t) = 0$ for every $t \in [a, b]$.

General Homogeneous Linear System

The following Corollary immediately follows from ***Abel's formula***.

Corollary

*A solution matrix $\mathbf{\Phi}(t)$ of (2) on $t \in [a, b]$ is a **fundamental matrix** of (2) on $t \in [a, b]$ if and only if $\det \mathbf{\Phi}(t) \neq 0$ for every $t \in [a, b]$.*

The ***initial value problem*** for the general ***linear homogeneous system*** satisfies:

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0,$$

where $A(t)$ is an $n \times n$ continuous matrix.

Theorem (Unique Solution)

*Assume that $\mathbf{\Phi}(t)$ is a **fundamental matrix solution** of (2) on $t \in [a, b]$. Then the **unique solution** of the **initial value problem** is given by:*

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0.$$

Example with $A(t)$ 1

Example: Consider the non-constant system of linear ***ODEs*** with $t > 0$:

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ \frac{4}{t^2} & -\frac{1}{t} \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(1) = \mathbf{x}_0 = \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix}. \tag{3}$$

Verify that the following are solutions to (3):

$$\mathbf{\Phi}_1(t) = \begin{pmatrix} t^{-2} \\ -2t^{-3} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}_2(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}.$$

Solution: From the ***system of ODEs*** we have

$$\dot{\mathbf{\Phi}}_1 = \begin{pmatrix} -2t^{-3} \\ 6t^{-4} \end{pmatrix} \quad \text{and} \quad A(t)\mathbf{\Phi}_1(t) = \begin{pmatrix} 0 & 1 \\ \frac{4}{t^2} & -\frac{1}{t} \end{pmatrix} \begin{pmatrix} t^{-2} \\ -2t^{-3} \end{pmatrix} = \begin{pmatrix} -2t^{-3} \\ 6t^{-4} \end{pmatrix}.$$

$$\dot{\mathbf{\Phi}}_2 = \begin{pmatrix} 2t \\ 2 \end{pmatrix} \quad \text{and} \quad A(t)\mathbf{\Phi}_2(t) = \begin{pmatrix} 0 & 1 \\ \frac{4}{t^2} & -\frac{1}{t} \end{pmatrix} \begin{pmatrix} t^2 \\ 2t \end{pmatrix} = \begin{pmatrix} 2t \\ 2 \end{pmatrix}.$$

Hence, it follows that $\mathbf{\Phi}_1(t)$ and $\mathbf{\Phi}_2(t)$ solve the ***system of ODEs***.

Example with $A(t)$ 2

Verify that $\mathbf{\Phi}(t) = [\mathbf{\Phi}_1(t), \mathbf{\Phi}_2(t)]$ forms a ***fundamental solution*** to (3).

Solution: We demonstrated that the columns of $\mathbf{\Phi}$ are solutions of (3), so the ***Corollary to Abel's Formula*** states that it suffices to verify that $\det \mathbf{\Phi}(t) \neq 0$.

$$\det \mathbf{\Phi}(t) = \det \begin{vmatrix} t^{-2} & t^2 \\ -2t^{-3} & 2t \end{vmatrix} = \frac{4}{t} \neq 0 \quad \text{for} \quad t > 0.$$

Find a ***fundamental solution***, $\mathbf{\Psi}(t)$ with $\mathbf{\Psi}(1) = I$.

Solution: Solve:

$$c_1\mathbf{\Phi}_1(1) + c_2\mathbf{\Phi}_2(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$d_1\mathbf{\Phi}_1(1) + d_2\mathbf{\Phi}_2(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Example with $A(t)$ 3

Equivalently,

$$c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad c_1 = c_2 = \frac{1}{2}.$$

and

$$d_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + d_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{or} \quad d_1 = -d_2 = -\frac{1}{4}.$$

It follows that another ***fundamental solution*** with $\mathbf{\Psi}(1) = I$ is given by:

$$\mathbf{\Psi}(t) = \begin{pmatrix} \frac{t^2+t^{-2}}{2} & \frac{t^2-t^{-2}}{4} \\ t - t^{-3} & \frac{t+t^{-3}}{2} \end{pmatrix}.$$

With this ***fundamental solution***, we readily obtain the ***unique solution*** to (3) given by:

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{x}_0 = \begin{pmatrix} \frac{t^2+t^{-2}}{2} & \frac{t^2-t^{-2}}{4} \\ t - t^{-3} & \frac{t+t^{-3}}{2} \end{pmatrix} \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix}$$

Example with $A(t)$ 4

How does we find a solution to (3) (without **Maple**)?

Solution: Earlier we showed how to transform ***2^{nd} order ODEs*** in ***systems of 1^{st} order ODEs***, so here we reverse the process.

The 1^{st} row of (3) gives $\dot{x}_1(t) = x_2(t)$, so $\dot{x}_2 = \ddot{x}_1 = \frac{4}{t^2}x_1 - \frac{1}{t}x_2 = \frac{4}{t^t}x_1 - \frac{1}{t}\dot{x}_1$, or

$$t^2\ddot{x}_1 + t\dot{x}_1 - 4x_1 = 0.$$

This is a ***Cauchy-Euler*** equation (solutions $x_1(t) = t^r$) with the ***auxiliary equation***:

$$r(r-1) + r - 4 = r^2 - 4 = 0 \quad \text{or} \quad r = \pm 2.$$

It readily follows that

$$x_1(t) = c_1t^{-2} + c_2t^2 \quad \text{and} \quad x_2(t) = -2c_1t^{-3} + 2c_2t.$$

Linear Nonhomogeneous System

Our work on ***Fundamental Solutions*** is a critical basis for solving the ***nonhomogeneous problem***.

Consider the general ***linear nonhomogeneous system*** given by:

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{g}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{4}$$

where both $A(t)$ and $\mathbf{g}(t)$ are continuous on some interval I.

Theorem (Variation of Constants Formula)

Let $\mathbf{\Phi}(t)$ be a ***fundamental matrix solution*** *of $\dot{\mathbf{x}} = A(t)\mathbf{x}$. Then the unique solution of (4) is given by:*

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0 + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds.$$

Proof for Variation of Constants

The ***variation of constants formula*** in our theorem states that given a ***particular solution***, then all other solutions only differ by the ***solution of the homogeneous equation***.

To find the ***particular solution***, assuming we know a ***fundamental matrix solution***, $\mathbf{\Phi}(t)$, to the homogeneous equation, we attempt $\mathbf{\Psi}_p$ of the form:

$$\mathbf{\Psi}_p(t) = \mathbf{\Phi}(t)\mathbf{v}(t),$$

with $\mathbf{v}(t)$ to be determined.

Differentiating gives:

$$\dot{\mathbf{\Psi}}_p(t) = \dot{\mathbf{\Phi}}(t)\mathbf{v}(t) + \mathbf{\Phi}(t)\dot{\mathbf{v}}(t) = A(t)\mathbf{\Phi}(t)\mathbf{v}(t) + \mathbf{g}(t).$$

With $\mathbf{\Phi}(t)$ solving the ***homogeneous problem***, the $\dot{\mathbf{\Phi}}(t)$ cancels $A(t)\mathbf{\Phi}(t)$, leaving

$$\mathbf{\Phi}(t)\dot{\mathbf{v}}(t) = \mathbf{g}(t).$$

Since $\mathbf{\Phi}(t)$ is nonsingular, integration yields the ***particular solution***:

$$\mathbf{v}(t) = \int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds \qquad \text{or} \qquad \mathbf{\Psi}_p(t) = \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds.$$

Constant Linear Nonhomogeneous System

For the case when we have a constant matrix A, then the ***linear nonhomogeneous system*** given by:

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{g}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{5}$$

where $\mathbf{g}(t)$ are continuous on some interval I has a simpler formulation.

Corollary (Variation of Constants Formula)

Let e^{At} be a ***fundamental matrix solution*** *of $\dot{\mathbf{x}} = A\mathbf{x}$. Then the unique solution of (5) is given by:*

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + \int_0^t e^{A(t-s)}\mathbf{g}(s)ds,$$

where $e^{-As} = \left(e^{As}\right)^{-1}$.

Example: Linear Nonhomogeneous System 1

Example: Consider the ***linear nonhomogeneous system*** given by:

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{g}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ t \end{pmatrix}, \qquad \text{with} \quad \mathbf{x}(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The matrix A is in our ***real Jordan canonical form***, which implies we can immediately write the ***fundamental matrix solution***:

$$e^{At} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

It is easy to see that the ***inverse*** satisfies:

$$e^{-At} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Example: Linear Nonhomogeneous System 2

Example: Next we compute the ***particular solution***:

$$\begin{aligned}
\mathbf{x}_p(t) &= e^{At}\int_0^t e^{-As}\mathbf{g}(s)ds, \\
&= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \int_0^t \begin{pmatrix} -s\,\sin(s) \\ s\,\cos(s) \end{pmatrix} ds \\
&= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} -\sin(t) + t\,\cos(t) \\ \cos(t) + t\,\sin(t) - 1 \end{pmatrix} \\
&= \begin{pmatrix} t - \sin(t) \\ 1 - \cos(t) \end{pmatrix}
\end{aligned}$$

With the ***initial condition***, the unique solution becomes:

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + x_p(t) = \begin{pmatrix} c_1\cos(t) + c_2\sin(t) + t - \sin(t) \\ -c_1\sin(t) + c_2\cos(t) + 1 - \cos(t) \end{pmatrix}.$$

Example 2: Linear Nonhomogeneous System 1

Example: Consider the ***linear nonhomogeneous system*** given by:

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{g}(t) = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 0 \\ t \end{pmatrix}, \qquad \text{with} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

It should be no surprise that Maple can readily solve this equation.

It is also apparent that the ***eigenvalues*** are $\lambda_1 = 3$ with ***algebraic*** and ***geometric multiplicity*** of **one** and ***associated eigenvector***, $\mathbf{v}_1 = [1, 0, 1]^T$

and $\lambda_2 = 2$ with ***algebraic*** and ***geometric multiplicities*** of **two** and **one**, respectively, and ***associated eigenvector***, $\mathbf{v}_2 = [1, 0, 0]^T$.

It follows that the ***Jordan canonical form*** is given by

$$J = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Example 2: Linear Nonhomogeneous System 2

Example: The previous slide gives the ***Jordan canonical form***, J, and with the help of Maple we obtain the ***transition matrix***, P, and its ***inverse***, P^{-1}:

$$J = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

The ***fundamental matrix solution*** follows readily from the ***Jordan canonical form***:

$$e^{Jt} = \begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & t\,e^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix}.$$

The ***fundamental matrix solution*** of the ***homogeneous*** part of the ***original ODE*** follows readily from:

$$e^{At} = Pe^{Jt}P^{-1} = \begin{pmatrix} e^{2t} & t\,e^{2t} & e^{3t} - e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}.$$

Example 2: Linear Nonhomogeneous System 3

Example: The ***variation of constants formula*** gives:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + \int_0^t e^{A(t-s)}\mathbf{g}(s)ds.$$

or

$$\begin{aligned}
\mathbf{x}(t) &= \begin{pmatrix} e^{2t} & t\,e^{2t} & e^{3t} - e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&\quad + \int_0^t \begin{pmatrix} e^{2(t-s)} & (t-s)e^{2(t-s)} & e^{3(t-s)} - e^{2(t-s)} \\ 0 & e^{2(t-s)} & 0 \\ 0 & 0 & e^{3(t-s)} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ s \end{pmatrix} ds.
\end{aligned}$$

Thus,

$$\mathbf{x}(t) = \begin{pmatrix} t\,e^{2t} + e^{3t} \\ e^{2t} \\ e^{3t} \end{pmatrix} + \int_0^t \begin{pmatrix} (1-s)e^{2(t-s)} + s\,e^{3(t-s)} \\ 0 \\ s\,e^{3(t-s)} \end{pmatrix} ds.$$

Example 2: Linear Nonhomogeneous System 4

Example: From before the *variation of constants formula* gives:

$$\mathbf{x}(t) = \begin{pmatrix} t\,e^{2t} + e^{3t} \\ e^{2t} \\ e^{3t} \end{pmatrix} + \int_0^t \begin{pmatrix} (1-s)e^{2(t-s)} + s\,e^{3(t-s)} \\ 0 \\ s\,e^{3(t-s)} \end{pmatrix} ds.$$

We let Maple perform these integrations, and the net result is:

$$\mathbf{x} = \begin{pmatrix} \frac{10}{9}e^{3t} + \left(t + \frac{1}{4}\right)e^{2t} + \frac{t}{6} - \frac{13}{36} \\ e^{2t} \\ \frac{10}{9}e^{3t} - \frac{t}{3} - \frac{1}{9} \end{pmatrix},$$

which is the *unique solution* to this example's *initial value problem*.

Linear Systems of ODEs
Fundamental Solution
General Linear System
Homogeneous System
Linear Nonhomogeneous System

Example 3: Linear Nonhomogeneous System 1

Example: Consider the non-constant, nonhomogeneous system of linear *ODEs* with $t > 0$:

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ \frac{4}{t^2} & -\frac{1}{t} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 10\,t^2 \\ 8 \end{pmatrix}, \qquad \mathbf{x}(1) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}. \tag{6}$$

In an earlier example, we demonstrated that a *fundamental solution* to the *homogeneous* part of (6) was given by:

$$\mathbf{\Phi}(t) = \begin{pmatrix} \frac{1}{t^2} & t^2 \\ -\frac{2}{t^3} & 2t \end{pmatrix}.$$

We also showed that $\det|\mathbf{\Phi}(t)| = \frac{4}{t}$ so it follows that:

$$\mathbf{\Phi}^{-1}(t) = \begin{pmatrix} \frac{t^2}{2} & -\frac{t^3}{4} \\ \frac{1}{2t^2} & \frac{1}{4t} \end{pmatrix}.$$

Linear Systems of ODEs
Fundamental Solution
General Linear System
Homogeneous System
Linear Nonhomogeneous System

Example 3: Linear Nonhomogeneous System 2

With the *fundamental solution*, $\mathbf{\Phi}(t)$, the *variation of constants formula* is applied giving:

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(1)\mathbf{x}_0 + \mathbf{\Phi}(t)\int_1^t \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds, \\
&= \begin{pmatrix} \frac{1}{t^2} & t^2 \\ -\frac{2}{t^3} & 2t \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 4 \\ 4 \end{pmatrix} + \begin{pmatrix} \frac{1}{t^2} & t^2 \\ -\frac{2}{t^3} & 2t \end{pmatrix}\int_1^t \begin{pmatrix} \frac{s^2}{2} & -\frac{s^3}{4} \\ \frac{1}{2s^2} & \frac{1}{4s} \end{pmatrix}\begin{pmatrix} 10\,s^2 \\ 8 \end{pmatrix} ds, \\
&= \begin{pmatrix} \frac{1}{t^2} + 3t^2 \\ -\frac{2}{t^3} + 6t \end{pmatrix} + \begin{pmatrix} \frac{1}{t^2} & t^2 \\ -\frac{2}{t^3} & 2t \end{pmatrix}\int_1^t \begin{pmatrix} 5s^4 - 2s^3 \\ 5 + \frac{2}{s} \end{pmatrix} ds, \\
&= \begin{pmatrix} \frac{1}{t^2} + 3t^2 \\ -\frac{2}{t^3} + 6t \end{pmatrix} + \begin{pmatrix} \frac{1}{t^2} & t^2 \\ -\frac{2}{t^3} & 2t \end{pmatrix}\begin{pmatrix} t^5 - \frac{t^4}{2} - \frac{1}{2} \\ 5t + 2\ln(t) - 5 \end{pmatrix}, \\
&= \begin{pmatrix} 2t^2\ln(t) + 6t^3 - \frac{5}{2}t^2 + \frac{1}{2t^2} \\ 4t\ln(t) + 8t^2 - 3t - \frac{1}{t^3} \end{pmatrix},
\end{aligned}$$

which gives the *unique solution* to our *initial value problem*.