1. Since 1973, the British Forestry Commission has surveyed for the presence of the American gray squirrel (*Sciurus carolinensis Gmelin*) and the native red squirrel (*Sciurus vulgaris L.*). From two consecutive years of data for 10 km square regions across Great Britain, data were collected on movement of the two types of squirrels. The transition matrix for red squirrels, gray squirrels, both, or neither in that order was given by

$$T = \begin{pmatrix} 0.8797 & 0.0382 & 0.0527 & 0.0008 \\ 0.0212 & 0.8002 & 0.0041 & 0.0143 \\ 0.0981 & 0.0273 & 0.8802 & 0.0527 \\ 0.0010 & 0.1343 & 0.0630 & 0.9322 \end{pmatrix}.$$

Find the equilibrium distribution of squirrels based on this transition matrix. Does this model suggest that the invasive gray species will significantly displace the native red squirrel over long periods of time?

2. An enclosed area is divided into four regions with varying habitats. One hundred tagged frogs are released into the first region. Earlier experiments found that on average the movement of frogs each day about the four regions satisfied the transition model given by

$$\begin{pmatrix} f_1(n+1) \\ f_2(n+1) \\ f_3(n+1) \\ f_4(n+1) \end{pmatrix} = \begin{pmatrix} 0.42 & 0.16 & 0.19 & 0.16 \\ 0.07 & 0.38 & 0.24 & 0.13 \\ 0.34 & 0.19 & 0.51 & 0.27 \\ 0.17 & 0.27 & 0.06 & 0.44 \end{pmatrix} \begin{pmatrix} f_1(n) \\ f_2(n) \\ f_3(n) \\ f_4(n) \end{pmatrix}.$$

a. Give the expected distribution of the tagged frogs after 1, 2, 5, and 10 days.

b. What is the expected distribution of the frogs after a long period of time? Which of the four regions is the most suitable habitat and which is the least suitable for these frogs?

c. These transitions are random events. Write a MatLab code for a Monte Carlo simulation of this experiment. (Show your code with comments to explain what the code is doing!) Run the experiment 1000 times, giving the mean and standard deviation of the distribution of frogs after 1, 2, 5, and 10 days. Compare these results to Part a.

3. a. Consider an animal that lives four years and reproduces annually. Animals that are 0-1 years old don't reproduce and only 40% ($s_1 = 0.4$) of them survive to the next year. Animals that are 1-2 years old produce on average $b_2 = 1.5$ offspring and 70% ($s_2 = 0.7$) of them survive to the next year. Animals 2-3 years old produce on average $b_3 = 2.2$ offspring and 75% ($s_3 = 0.75$) of them survive to the next year. Finally, animals 3-4 years old produce $b_4 = 3.4$ offspring. Create a model using a Leslie matrix, L, of the form:

$$P_{n+1} = LP_n.$$

Find the steady-state percentage of each age group. Determine how long it takes for this population to double after it has reached its steady-state distribution.

b. Assume that a fraction of 2-4 year olds are harvested. That is, the survival rates s_2 and s_3 are reduced. If the survival rates are reduced by a fraction α, so that the survival rate of 1-2 year olds is $0.7\,\alpha$ and the survival rate of 2-3 year olds is $0.75\,\alpha$. Determine the value of α that leaves the population at a constant value. For this value of α (to at least 3 significant figures), if there are 550 mature (3-4 year olds), then determine the total population and number in each population age group. How many animals are harvested annually under these conditions?

4. An age-structured population of birds was surveyed over 4 years. The researchers determined the number of birds in each age class for each of the 4 years and found out how many nestlings fledged from each of the different age classes each year. The researchers divided the population of birds into the birds 0-1 years old, 1-2 years old, and those that are older. This age-structured population forms a Leslie model of the following form:

$$\begin{pmatrix} P_1(n+1) \\ P_2(n+1) \\ P_3(n+1) \end{pmatrix} = \begin{pmatrix} 0 & b_2 & b_3 \\ s_{12} & 0 & 0 \\ 0 & s_{23} & s_{33} \end{pmatrix} \begin{pmatrix} P_1(n) \\ P_2(n) \\ P_3(n) \end{pmatrix}.$$

a. The table below shows how many birds in each age class survived to the next year (and gives the total number of birds that fledged). The researchers determined that the survival of the 1-2 year old birds is roughly equal to the survival of the older birds. Thus, we can assume that $s_{23} = s_{33}$. Use the data below to compute the average values for each of the survival parameters s_{12} and $s_{23} = s_{33}$.

Bird Age	Year 1	Year 2	Year 3	Year 4
0-1	175	237	258	311
1-2	42	59	89	92
older	97	104	128	145

They also collected data on the success rate of nesting of each of the different age classes of birds. The table below shows the number of fledglings raised by each of the age classes over the 4 year period. (Note that these columns total to the number of 0-1 year old birds the next year.) Use the data below to compute the average birth rates for each of the age classes b_2 and b_3. (One year old birds of this species don't nest.)

Bird Age	Year 1	Year 2	Year 3	Year 4
1-2	38	47	66	74
older	199	211	245	293

b. Write the Leslie matrix for this species of bird using the average values computed above (to **4 significant figures**). Use your Leslie matrix to estimate the population of each of the age classes for the next 3 years. (Use the last surveyed data as your starting point for this simulation.)

c. Find the eigenvalues and eigenvectors for this model, then give the limiting percent population in each of the age classes. What is the approximate annual rate of growth for this species of bird and how long would it take for the total population to double?

5. Enzymes are critical to sustaining life. Enzymes are capable of very rapidly converting one substance to another at room temperature. Many enzymes are what are known as Michaelis-Menten enzymes, which have simple first order kinetics. Let S_1 be a substrate, S_2 be an enzyme, S_3 be the enzyme-substrate complex, and S_4 be the resultant product. The enzymatic transformation of S_1 into S_4 satisfies the chemical reactions:

$$
\begin{aligned}
S_1 + S_2 &\overset{c_1}{\longrightarrow} S_3, \\
S_3 &\overset{c_2}{\longrightarrow} S_1 + S_2, \\
S_3 &\overset{c_3}{\longrightarrow} S_2 + S_4.
\end{aligned}
$$

a. *E. coli* is a bacterium with a volume, $V = 7 \times 10^{-19} l$, that uses enzymes to convert different chemical species into other metabolic products. In this problem, we assume that a cell begins with $S_1 = 500$ molecules of subtrate, $S_2 = 50$ molecules of enzyme, and $S_3 = S_4 = 0$ molecules of complex and product, respectively. The rate constants satisfy:

$$c_1 = 0.002, \qquad c_2 = 6 \times 10^{-5}, \qquad \text{and} \qquad c_3 = 0.08.$$

Develop a Gillespie Algorithm for these Michaelis-Menten enzyme reactions. Show your code and present **2** distinct simulations of the time series for $t \in [0, 200]$, showing all **4** chemical species. Briefly describe what you see from your simulations and explain what is happening with the different chemical species.

b. The reactions above can be written into a system of **4** ordinary differential equations. Show your system of differential equations and simulate for $t \in [0, 200]$, using the same reaction constants above. Show a graph of this ODE simulation overlayed with one from the Gillespie stochastic simulation algorithm for the same reactions. Briefly discuss the similarities and differences between these two simulation methods.

c. The system of ODEs for the Michaelis-Menten reactions are often simplified using the quasi-steady state approximations. By assuming that the reactions forming the complex, S_3 occur rapidly compared to the production of the product, S_4, the system of ODEs in Part b can be written as:

$$
\begin{aligned}
\frac{ds_1}{dt} &= -\frac{V_m s_1}{K_m + s_1} \\
\frac{ds_4}{dt} &= \frac{V_m s_1}{K_m + s_1},
\end{aligned}
\tag{1}
$$

where s_1 and s_4 are the concentrations of the substrate, S_1, and the product, S_4. The constants, V_m and K_m, are kinetic constants.

Simulate the ODE in Part b and collect data at times, $t_n = 10n, n = 0, 1, ..., 20$, for $S_1(t_n)$ and $S_4(t_n)$. Use the least squares techniques from earlier in the class to fit the System (1) to these data by adjusting the constants, V_m and K_m. Give the values of the constants, V_m and K_m, and the sum of square errors. Create a graph of the substrate and product from simulations of the systems in Parts b and c. Briefly discuss the similarities and differences between these two simulations.

d. The computations above are all using the nonstandard units of *molecules/E. coli* (with dimensionless time). Rewrite the initial conditions, and the important kinetic constants, V_m and K_m, in standard units of M (moles/liter). Also, if the nondimensional time units above are in msec (milliseconds), then include the conversion to seconds for any constant requiring time.