All answers need to be written in a clear, succinct manner. Write a brief paragraph summarizing the answers to the problems with each answer clearly stated in a sentence. Supporting graphs should be provided when asked for, but you should not include printouts of spreadsheets. You can create an appendix to a problem, but that should only include significant material to back up your answers.

1. Work the SIR Model problem in WeBWorK, including written work and graphs in your assignment.

2. a. An alternate SIR Model includes births and deaths. For simplicity, we assume that the birth rate and death rate are the same so that total population stays the same. We also assume that we are analyzing a disease that is not vertically transmitted, so all births go into the susceptible class. The SIR model now satisfies the discrete dynamical system given by:

$$\begin{aligned} S_{n+1} &= S_n - \frac{\beta}{N} S_n I_n + b(I_n + R_n), \\ I_{n+1} &= I_n(1 - \gamma - b) + \frac{\beta}{N} S_n I_n, \\ R_{n+1} &= R_n(1 - b) + \gamma I_n, \end{aligned} \tag{1}$$

where β is the contact rate, γ is the rate of recovery, and b is the birth and death rate. The assumption is that the population is constant, so

$$N = S_n + I_n + R_n.$$

Use this information to reduce the model to two difference equations in S_n and I_n. Find all equilibria for this model. Give a condition relating the parameters β, γ, and b that implies all equilibria are non-negative.

b. Linearize the model with the two difference equations that you found above. Give the general Jacobian matrix, then find the eigenvalues for the disease free equilibrium, $I_e = 0$. The *basic reproduction number*, R_0, is given by

$$R_0 = \frac{\beta}{\gamma + b}.$$

How many nonnegative equilibria are there when $R_0 < 1$? How about when $R_0 > 1$? Discuss the local stability of the disease free equilibrium for $R_0 < 1$ and $R_0 > 1$.

c. Assume $N = 100$ is the constant total population. Let $\beta = 0.3$, $\gamma = 0.2$, and $b = 0.2$ with initial populations $S_0 = 70$ and $I_0 = 30$. Simulate the two difference equation model for $n \in [0, 25]$. Show a graph of your simulation (both S_n and I_n) and describe what you observe. Find R_0. What does this simulation say about this particular disease?

d. Find all equilibria for this system (possibly including ones with negative populations). Linearize about the equilibria. Find the eigenvalues for the linearized system at each of the equilibria, and discuss the behavior of the linearized system near those equilibria. Does this analysis support your observations in Part c?

e. Repeat the steps you did in Parts c and d with $\beta = 0.8$, $\gamma = 0.1$, and $b = 0.1$.