Math 636 - Mathematical Modeling

Age-Structured Models

Joseph M. Mahaffy,
⟨jmahaffy@mail.sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://jmahaffy.sdsu.edu

Fall 2018

Outline

Leslie Models 1

Leslie Models: Earlier examined discrete models combining all members of the population in a single group.

- Reproduction and survival depend highly upon age.
- Improved population models include some ***age-structure*** in the model.
- The simplest ***age-structured model*** is the ***Leslie model*** (Patrick H. Leslie (1900-1974)), developed in the 1940s.
- This ***discrete population model***, in which the population of one sex (usually females) is divided into ***discrete age classes***.
- The population is closed to migration and only considers ***births*** and ***deaths*** amongst the ages classes or life stages.
- This model studies the growth of a population and determines the relative size of each of the age classes.

Leslie Models

Leslie Models: The population, $\mathbf{X}_n$, is a vector, which varies discretely in time.

- Let L be the Leslie matrix with $\mathbf{X}_n = [x_1, x_2, ..., x_m]_n^T$, the population vector at time n divided into m age classes, then the Leslie model is:

$$\mathbf{X}_{n+1} = L\mathbf{X}_n.$$

- Let b_i be the ***per capita birth rate*** of the i^{th} class into the first class, and s_i be the ***survival rate*** of individuals of class i at time n into class $i+1$ at time $n+1$, then the ***Leslie matrix*** takes the form:

$$L = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_m \\ s_1 & 0 & 0 & \cdots & 0 \\ 0 & s_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & s_{m-1} & s_m \end{pmatrix}.$$

- $s_m = 0$ if the older classes not considered in the population or all the population dies after class m.
- The survival $s_m \neq 0$, if the oldest class includes all the oldest individuals past the age class of $m-1$.

Analysis of the Leslie Model

Analysis of the Leslie Model: Assume that the matrix L has a complete set of ***eigenvalues***, λ_i, $i = 1, ..., m$ with associated eigenvectors, ξ_i spanning $\mathbb{R}^m$, then the ***initial population*** can be written:

$$\mathbf{X}_0 = \sum_{i=0}^{m} c_i \xi_i,$$

for some constants, c_i.

The ***Leslie model*** gives:

$$\mathbf{X}_1 = L\mathbf{X}_0 = \sum_{i=0}^{m} c_i \lambda_i \xi_i,$$

The ***Leslie model*** follows a ***Markov process***, so the n^{th} population, $\mathbf{X}_n$, is given by:

$$\mathbf{X}_n = L^n \mathbf{X}_0 = \sum_{i=0}^{m} c_i \lambda_i^n \xi_i, \qquad n = 1, ...$$

Analysis of the Leslie Model

Analysis of the Leslie Model: Assume that the ***Leslie matrix***, L, has a ***single dominant eigenvalue***, λ_1.

If

$$|\lambda_1| > \max_{i=2,..,m} |\lambda_i|,$$

then for n sufficiently large $|\lambda_1|^n \gg |\lambda_i|^n$ for $i = 2, ..., m$.

Asymptotically, it follows that

$$\mathbf{X}_n \simeq c_1 \lambda_1^n \xi_1.$$

It follows that the ***population grows*** or ***decays*** much like a ***Malthusian growth model*** with the exponential growth, λ_1 and the age classes having a distribution of ξ_1.

Example of U. S. Population Growth 1

Example of U. S. Population Growth: The female population is divided into **3** age groups of 20 years.

A detailed study of these age groups give the fertility of each group and the survival of each age group.

Let x_1 represent females age 0-20, x_2 represent females age 20-40, and x_3 represent females age 40-60, then a ***Leslie model*** for the dynamics of these age classes in the U. S. is given by:

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \\ x_3(n+1) \end{pmatrix} = \begin{pmatrix} 0.4271 & 0.8498 & 0.1273 \\ 0.9924 & 0 & 0 \\ 0 & 0.9826 & 0 \end{pmatrix} \begin{pmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \end{pmatrix}.$$

Example of U. S. Population Growth

Example (cont): The ***Leslie matrix*** satisfies:

$$L = \begin{pmatrix} 0.4271 & 0.8498 & 0.1273 \\ 0.9924 & 0 & 0 \\ 0 & 0.9826 & 0 \end{pmatrix}.$$

The ***Leslie matrix*** implies:

- The typical female age 0-20 produces 0.4271 female offspring in a twenty year period.
- The typical 20-40 year old females produce the most female offspring (peak fertility) with a per capita rate of 0.8498
- Females age 40-60 have significantly fewer female offspring at only 0.1273 per woman in those older years.
- Survival rate to successive classes is very high with 0-20 year olds having a 99.24% chance of making it into the 20-40 age group
- The 20-40 year olds having a 98.26% chance of surviving into the 40-60 year old age class.
- This model ignores the contribution of any female over 60 (though it would not be hard to include them in a 40 and older class and not have the $L_{33} = 0$).

Example of U. S. Population Growth: Analysis

Analysis of Example: An ***eigenvalue analysis*** of the ***Leslie matrix***, L, gives:

$$\lambda_1 = 1.2093, \qquad \lambda_2 = -0.6155, \qquad \text{and} \qquad \lambda_3 = -0.1668.$$

- The ***dominant eigenvalue*** is $\lambda_1 = 1.2093$, which implies that the population growth is approximately 21% in a twenty year period.
- This is consistent with the ***Malthusian growth rate*** obtained earlier for the U. S. population from the census data in the latter part of the 20^{th} century.
- The ***normalized eigenvector associated*** with λ_1 is

$$\xi_1 = [0.4020, 0.3299, 0.2681]^T.$$

- This shows that the current distribution of females should be approximately 40.2% in the 0-20 age group, 33% in the 20-40 age group, and 26.8% in the 40-60 age group.
- From the U. S. Census Bureau, the data from 2010 gives the current distribution of females as 33.5% in the 0-20 age group, 34.5% in the 20-40 age group, and 32.0% in the 40-60 age group (with these classes composing 79.8% of all women in the country).
- What causes the discrepancies between the Leslie model and the actual data?

Example of Loggerhead Turtles 1

Loggerhead Turtles: Crowder [1] used a modified version of the Leslie matrix model to predict the impact of turtle excluder devices (TEDs) on the populations of ***loggerhead sea turtles***.

- Crowder *et al.* updated an earlier model to examine how enforcing the use of TEDs might impact the ***loggerhead sea turtle*** population.
- Studies have shown TEDs can be up to 97% effective in preventing mortality of large turtles.
- A high percentage of dead sea turtles on beaches can be shown to have died from shrimp fishing with nets.
- Crowder *et al.* used a ***stage-based population model***, which is closely related to ***Leslie models***.
- ***Model analysis*** showed different scenarios, which could result in population recoveries of this vanishing species.

[1] Larry B. Crowder, Deborah T. Crouse, Selina S. Heppell, and Thomas H. Martin, (1994) Predicting the Impact of Turtle Excluder Devices on Loggerhead Sea Turtle Populations, *Ecological Applications*, **4**, 437-445.

Example of Loggerhead Turtles

Loggerhead Turtles: The earlier model of Crowder *et al.* was a detailed study of data on ***loggerhead sea turtles***.

- The study carefully measured survivorship and fecundity of different age groups of ***loggerhead turtles***.
- The original model had **7** age groups, which were reduced to **5** age groups.
- Adult turtles lay their eggs on sandy beaches.
- Only about 1% of turtles surviving the first year reach sexual maturity around age 20 and can reproduce.
- Much of their life cycle remains a mystery, complicating conservation efforts.

Example of Loggerhead Turtles

Loggerhead Turtle Example: The model is based on a **5 *stage-classified model***:

$$\mathbf{X}_{n+1} = A\mathbf{X}_n,$$

with

$$A = \begin{pmatrix} P_1 & F_2 & F_3 & F_4 & F_5 \\ G_1 & P_2 & 0 & 0 & 0 \\ 0 & G_2 & P_3 & 0 & 0 \\ 0 & 0 & G_3 & P_4 & 0 \\ 0 & 0 & 0 & G_4 & P_5 \end{pmatrix},$$

where P_i is the probability of surviving and remaining in the same stage, G_i is the probability of surviving and growing to the next stage, F_i is the stage-specific reproductive output, and n is in years.

$\mathbf{X}_n$ gives **5** stages of turtle development:

Stage	Description	Duration (yrs)
1	Eggs/hatchlings	1
2	Small Juveniles	7
3	Large Juveniles	8
4	Subadults	6
5	Adults	> 22

Example of Loggerhead Turtles

Loggerhead Turtle Example: The **5** ***stage-classified model*** from the data gives:

$$\mathbf{X}_{n+1} = \begin{pmatrix} 0 & 0 & 0 & 4.665 & 61.896 \\ 0.675 & 0.703 & 0 & 0 & 0 \\ 0 & 0.047 & 0.657 & 0 & 0 \\ 0 & 0 & 0.019 & 0.682 & 0 \\ 0 & 0 & 0 & 0.061 & 0.8091 \end{pmatrix} \mathbf{X}_n,$$

which shows the primary reproduction is from adult turtles, and the adult turtles have a fairly high survivorship.

As with the ***Leslie model***, the ***eigenvalues*** and ***eigenvectors*** are found with the ***dominant eigenvalue*** and its ***normalized eigenvector*** being:

$$\lambda_1 = 0.9516 \qquad \text{and} \qquad \xi_1 = \begin{pmatrix} 0.2385 \\ 0.6477 \\ 0.1033 \\ 0.007284 \\ 0.003118 \end{pmatrix}.$$

This model suggests a **5% decline** in population each year.

Example of Loggerhead Turtles

Loggerhead Turtle Example: If the TEDs could increase ***survivorship*** of larger turtles by 20%, then the **5** ***stage-classified model*** becomes:

$$\mathbf{X}_{n+1} = \begin{pmatrix} 0 & 0 & 0 & 4.665 & 61.896 \\ 0.675 & 0.703 & 0 & 0 & 0 \\ 0 & 0.047 & 0.719 & 0 & 0 \\ 0 & 0 & 0.021 & 0.723 & 0 \\ 0 & 0 & 0 & 0.071 & 0.847 \end{pmatrix} \mathbf{X}_n.$$

This ***stage-classified model*** has a ***dominant eigenvalue***, λ_1, and its ***normalized eigenvector*** being:

$$\lambda_1 = 1.00854 \qquad \text{and} \qquad \xi_1 = \begin{pmatrix} 0.2714 \\ 0.5995 \\ 0.1168 \\ 0.008588 \\ 0.003774 \end{pmatrix}.$$

This model suggests a **slight increase** in population each year.

SDSU