

# Math 636 - Mathematical Modeling

## Discrete Modeling U. S. Population

Joseph M. Mahaffy,  
([jmahaffy@sdsu.edu](mailto:jmahaffy@sdsu.edu))

Department of Mathematics and Statistics  
Dynamical Systems Group  
Computational Sciences Research Center  
San Diego State University  
San Diego, CA 92182-7720  
<http://jmahaffy.sdsu.edu>

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## Outline

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## United States Census

## Census Data

### United States Census

- Constitution requires census every 10 years
- Census used for budgeting federal payments and representation in Congress
- Process can be politically charged
- Accurately predicting demographic data are important for planning communities in the future
- Calculations for the future populations uses sophisticated mathematical models
- Models are constantly improved and revised

### Census Data

|      |            |      |             |      |             |
|------|------------|------|-------------|------|-------------|
| 1790 | 3,929,214  | 1870 | 39,818,449  | 1950 | 150,697,361 |
| 1800 | 5,308,483  | 1880 | 50,189,209  | 1960 | 179,323,175 |
| 1810 | 7,239,881  | 1890 | 62,947,714  | 1970 | 203,302,031 |
| 1820 | 9,638,453  | 1900 | 76,212,168  | 1980 | 226,545,805 |
| 1830 | 12,866,020 | 1910 | 92,228,496  | 1990 | 248,709,873 |
| 1840 | 17,069,453 | 1920 | 106,021,537 | 2000 | 281,421,906 |
| 1850 | 23,191,876 | 1930 | 122,775,046 | 2010 | 308,745,538 |
| 1860 | 31,443,321 | 1940 | 132,164,569 |      |             |

History of the Census Bureau shows that these numbers are collected in various ways, including information only from the heads of households, the Bureau depending on only 183 clerks for over 30M people, failure to count slaves, and other irregularities such as recent undercounting minorities.



## Growth Rate of U. S.

### Growth Rate in Early U. S.

The growth rate for the decade of 1790-1800

$$\frac{\text{Population in 1800}}{\text{Population in 1790}} = \frac{5,308,483}{3,292,214} = 1.351$$

The growth rate for the decade of 1800-1810

$$\frac{\text{Population in 1810}}{\text{Population in 1800}} = \frac{7,239,881}{5,308,483} = 1.364$$

The growth rate for the decade of 1810-1820

$$\frac{\text{Population in 1820}}{\text{Population in 1810}} = \frac{9,638,453}{7,239,881} = 1.331$$

These growth rates are relatively **constant**, averaging 0.349/decade, and are typical of agrarian societies.



## Malthusian Growth Model

### Malthusian Growth Model

- Simplest growth model uses a constant rate of growth,  $r$
- Start with the population in 1790,  $P_0$
- Population in the next decade is current population plus the population times the average growth rate

$$P_{n+1} = P_n + rP_n = (1 + r)P_n$$

- Sequence of predicted populations based solely on population from preceding population
- The **Malthusian Growth Model** is easily solved to give:

$$P_n = P_0(1 + r)^n.$$



## Malthusian Growth Model

**Table for U. S. Population** (early years), assuming a **constant growth rate** of 0.349

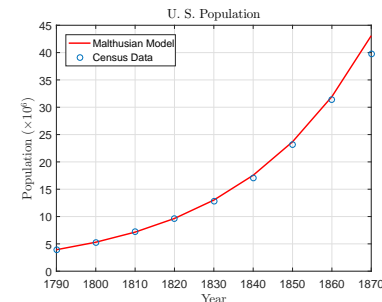
| Year | Census     | Model $P_{n+1} = 1.349P_n$ | % Error |
|------|------------|----------------------------|---------|
| 1790 | 3,929,214  | 3,929,214                  |         |
| 1800 | 5,308,483  | 5,300,510                  | -0.15   |
| 1810 | 7,239,881  | 7,150,388                  | -1.24   |
| 1820 | 9,638,453  | 9,645,873                  | 0.08    |
| 1830 | 12,866,020 | 13,012,282                 | 1.14    |
| 1840 | 17,069,453 | 17,553,569                 | 2.84    |
| 1850 | 23,191,876 | 23,679,765                 | 2.10    |
| 1860 | 31,433,321 | 31,944,002                 | 1.59    |
| 1870 | 39,818,449 | 43,092,459                 | 8.22    |

The **Malthusian Growth Model** tracks early U. S. population well until the Civil war.



## Malthusian Growth Model

Graph of the **Malthusian Growth Model** and **Census Data** for the U. S.



**Error** remains small until 1870, as U. S. is primarily an **agrarian society** with a constant growth rate around **35%**

**Civil War** dramatically increased the death rate, but more significantly, society moved into the **industrial revolution**



## Programming Discrete Models

**Programming Discrete Models:** Discrete dynamical models are one of the easiest models to program:

$$x_{n+1} = f(x_n)$$

- Spreadsheet software
  - **Excel** is designed for *discrete dynamical models*
  - The *pull-down* or *updating* feature automatically uses the value in the cell above to insert into the “updating function” in the new cell
- **MatLab** or other programming languages
  - *Discrete dynamical models* are iterated processes, which naturally program with **for** loops
  - In **MatLab** if we want to iterate the model above  $N$  times creating a solution vector  $x$  from a given function  $f(x)$ , we write:

```
1 x = [x0];
2 for i = 1:N
3     x0 = f(x0);
4     x = [x, x0];
5 end
```

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Joseph M. Mahaffy, (jmahaffy@sdsu.edu)

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## Malthusian Growth Model

```
1 clear
2 xlabel = 'Year';
3 ylabel = 'Population ($\times 10^6$)';
4 mytitle = 'U. S. Population';
5 pcensus = [3.929 5.308 7.240 9.638 12.866 17.069 ...
            23.192 31.433 39.818];
6 tt = [1790]; pmal = [3.929];
7 for i=1:8
8     t = tt(i) + 10;
9     mal = pmal(i) + 0.349*pmal(i);
10    tt = [tt,t];
11    pmal = [pmal,mal];
12    formatSpec = 'Year is %d, Population is %8.3f ...
                 M\n';
13    fprintf(formatSpec,t,mal)
14 end
```

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## Malthusian Growth Model

```
15 plot(tt,pmal,'r-','LineWidth',1.5);
16 hold on
17 plot(tt,pcensus,'o','MarkerSize',6);
18 grid
19 legend('Malthusian Model','Census ...
        Data','Location','Northwest');
20
21 ax = [1790 1870 0 50];
22 fontlabs = 'Times New Roman';
23 xlabel(xlab,'FontSize',14,'FontName',fontlabs,'interpreter','
24 ylabel(ylab,'FontSize',14,'FontName',fontlabs,'interpreter','
25 title(mytitle,'FontSize',16,'FontName','Times New ...
        Roman','interpreter','latex');
26 set(gca,'FontSize',12);
27 print -depsc uspopmal.eps
```

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## Analysis of the Growth Rate

## Variation in Growth Rate

- Assume this Malthusian growth model were extended
  - In 1920, model predicts 192,365,343 (population in 1960s), which is 82% too high
  - In 1970, model predicts 859,382,645, which is 323% too high
- *Growth rate of U. S.* has generally decreased over the *Census history*
  - Average growth rate over census history is 22.3%
  - In 1800, 1900, and 2000, the growth rates were 36.4%, 21.0%, and 9.71%, showing the general decline
  - The Great Depression gave lowest growth rate of 7.2%
  - Obvious *time variation* in growth rate

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## Modeling the Growth Rate

## Modeling the Growth Rate

- Malthusian growth model is too simplistic except early population growth with plenty of resources
- Crowding and limited resources are usually next dominant term in modeling (*logistic growth*)
  - Crowding/squalor leads to declining birth rates and increased death rates
  - Improved health decreases death rates and significantly drops birth rates (greatest birth rate decline comes from educating women)
- Clearly, human populations have important contributions from time varying historical, societal, and technological changes (*temporal variation*)
- Animal population models usually ignore these time-varying factors (*autonomous*)



## General Discrete Dynamical Population Model

## General Discrete Dynamical Population Models (cont)

- If the population at the next discrete time period depends on the current population and time, then given initial population,  $P_0$ ,

$$P_{n+1} = f(t_n, P_n),$$

for some function  $f(t, P)$

- This *first order difference equation* is *nonautonomous*
- The inclusion of time complicates *analysis* of this type of model
- Human populations have significant time-varying changes, which make this type of model preferable



## General Discrete Dynamical Population Model

## General Discrete Dynamical Population Models

- If the population at the next discrete time period depends only on the current population, then given initial population,  $P_0$ ,

$$P_{n+1} = f(P_n),$$

for some function  $f(P)$

- This *first order difference equation* is *autonomous*
- *Malthusian growth* uses a *linear growth* with  $f(P) = P + rP$ , while *logistic growth* uses a *quadratic growth* with  $f(P) = P + rP(1 - P/M)$
- Many different nonlinear forms of  $f(P)$  are used with different interpretations and mathematical behaviors
- Animal populations often use these models



## General Discrete Dynamical Population Model

## Malthusian Growth Model

**Malthusian Growth Model:** This model has the solution:

$$P_n = (1 + r)^n P_0,$$

which has *two parameters*,  $r$  and  $P_0$ .

- If a *nonlinear least squares best fit* of this model to the U. S. census data is applied, the best fitting model is

$$P_n = 16.35(1.1460)^n,$$

where  $P_0 = 16.35$  and  $r = 0.1460$

- The SSE = 2875
- This model poorly fits the early and late census data, but it is extremely simple with only two parameters



## Malthusian Growth Model - MatLab

Use **MatLab** to find the nonlinear least squares fit to the census data (tdata, pdata) with  $p_0 = [P_0, r]$

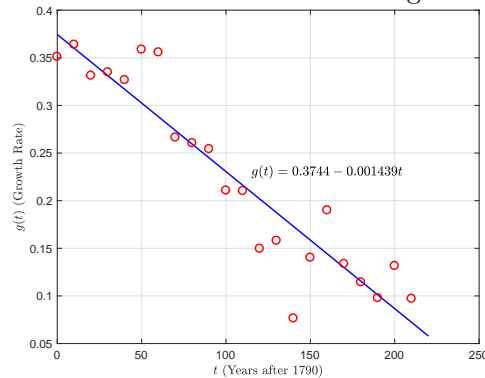
```
1 function J = mal_lst(p0,tdata,pdata)
2 % Least Squares fit to Malthusian Growth
3 N = length(tdata);
4 p = p0(1);
5 pop = [p];
6 err = [pdata(1) - p];
7 for i = 2:N % Malthusian iteration
8     p = p*(1+p0(2));
9     pop = [pop,p];
10    err = [err, pdata(i) - p];
11 end
12 J = err*err'; % Sum of square errors
13 end
```

p1 = fminsearch(@mal\_lst,p0,[],tdata,pdata)



## Linear Growth Rate

**Linear Growth Rate:** Directly from the Census data the *best fitting linear model* is found from the decade growth rates.



## Nonautonomous Malthusian Growth Model

**Linear Growth Rate:** Earlier saw that over U. S. history the growth rate generally declined

- Human populations vary with technology and many other time varying factors
- Modify the *Malthusian growth model* to include a *linear growth rate*:

$$P_{n+1} = P_n + (a + bt_n)P_n$$

- Begin by computing the growth rate from all of the census data
- Find the best fitting line through these growth rates
- The best fitting line satisfies:

$$r(t) = 0.37441 - 0.014390n,$$

where  $n$  is in decades after 1790

- If this model is fit to the census data by adjusting  $P_0$ , then the best  $P_0 = 3.758$  with a SSE = 740.3



## Nonautonomous Malthusian Growth Model

**Nonautonomous Malthusian Growth Model:** Alternately, consider

$$P_{n+1} = P_n + (a + bt_n)P_n,$$

with initial population,  $P_0$

- This model has **3** parameters
- Run a *nonlinear least squares best fit* for the parameters,  $a$ ,  $b$ , and  $P_0$
- The best fitting *nonautonomous Malthusian growth model* satisfies:

$$P_{n+1} = (1.3058 - 0.009674n)P_n,$$

with  $P_0 = 6.305$  and  $n$  is in decades after 1790

- This model fits the census data quite well, and the SSE = 326.8
- Over the range of the census data, this model does better, but it is further off initially



## Nonautonomous Malthusian Growth Model - MatLab

Use **MatLab** to find the nonlinear least squares fit to the census data (tdata, pdata) with  $p0 = [P_0, a, b]$

```

1 function J = nonmal_lst(p0,tdata,pdata)
2 % Least Squares fit to Nonautonomous Malthusian ...
   Growth
3 N = length(tdata);
4 p = p0(1);
5 pop = [p];
6 err = [pdata(1) - p];
7 for i = 2:N % Nonautonomous iteration
8     p = p*(1 + p0(2) + p0(3)*(i-1));
9     pop = [pop,p];
10    err = [err, pdata(i) - p];
11 end
12 J = err*err'; % Sum of square errors
13 end

```

```
p1 = fminsearch(@nonmal_lst,p0,[],tdata,pdata)
```

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Joseph M. Mahaffy, (jmahaffy@sdsu.edu)

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## Logistic Growth Model

**Logistic Growth Model** is an *autonomous model*, which is often applied to animal populations

This model has the Malthusian linear growth and includes a negative quadratic term for crowding:

$$P_{n+1} = P_n + rP_n \left(1 - \frac{P_n}{M}\right),$$

with initial population,  $P_0$

- This model has **3** parameters, including the initial population,  $P_0$
- **Nonlinear least squares program** gives the best fitting parameters,  $r$ ,  $M$ , and  $P_0$
- The best fitting **logistic growth model** satisfies  $P_0 = 8.575$  and:

$$P_{n+1} = P_n + 0.2245P_n \left(1 - \frac{P_n}{451.7}\right),$$

- This model fits the census data quite well, and the SSE = 557.4
- This model is only a slightly worse fit than the **nonautonomous Malthusian growth model**

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Joseph M. Mahaffy, (jmahaffy@sdsu.edu)

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## Beverton-Holt Model

**Beverton-Holt Model**, commonly used in ecology, is an *autonomous model*, which uses a different **nonlinear updating function**:

$$P_{n+1} = \frac{aP_n}{1 + \frac{P_n}{b}},$$

with initial population,  $P_0$

This nonlinear function remains positive for all  $P_n$ , unlike the **logistic growth updating function**

- This model has **3** parameters, including the initial population,  $P_0$
- **Nonlinear least squares program** gives the best fitting parameters,  $a$ ,  $b$ , and  $P_0$
- The best fitting **logistic growth model** satisfies  $P_0 = 8.261$  and:

$$P_{n+1} = \frac{1.23065P_n}{1 + \frac{P_n}{2110.5}},$$

- This model fits the census data quite well, and the SSE = 519.5
- This model compares well to the **logistic growth model**, despite a very different **updating function**

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Joseph M. Mahaffy, (jmahaffy@sdsu.edu)

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## Autonomous Models – MatLab

The **nonlinear least squares best fit** of the **logistic** and **Beverton-Holt models** use almost identical algorithms to the **nonautonomous Malthusian growth model** with **3** parameters to fit; however, the iteration formulas vary with the model

```

7 for i = 2:N % Logistic iteration
8     p = p + p0(2)*p*(1 - p/p0(3));

```

```

7 for i = 2:N % BevertonHolt iteration
8     p = p0(2)*p/(1 + p/p0(3));

```

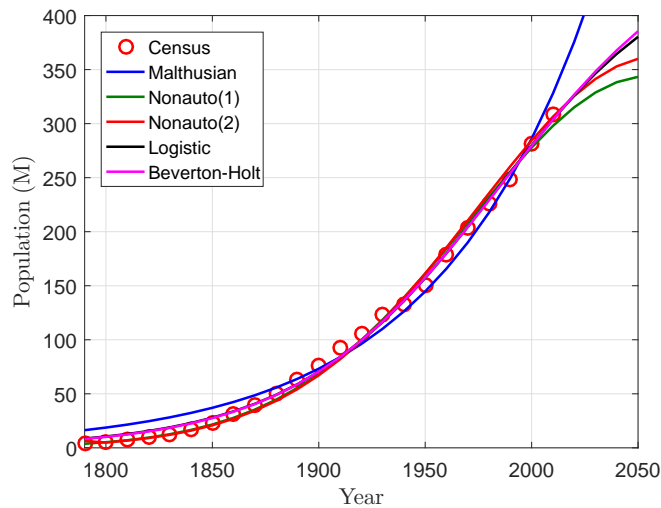
The use of `fminsearch` is also very similar

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## Graph of U. S. Population Models



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## General Discrete Population Models

**Discrete population models** are generally written in one of two forms

A **discrete growth population model** is written in the form:

$$P_{n+1} = P_n + g(t_n, P_n),$$

which says the population at the new discrete time,  $n + 1$ , is equal to the old population,  $P_n$ , plus the **net growth**,  $g(t_n, P_n)$ , which depends on the previous time,  $t_n$ , and the population at the previous time,  $P_n$ .

A more general **discrete population model** is written in the form:

$$P_{n+1} = f(t_n, P_n),$$

which says the population at the new discrete time,  $n + 1$ , is equal to some **updating function**,  $f(t_n, P_n)$ , which depends on the previous time,  $t_n$ , and the population at the previous time,  $P_n$ .

Most of our **analysis** will consider **autonomous updating functions** of the form,  $f(P_n)$ , having no time dependence.

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## Analysis of Autonomous Models

We begin the basics of **qualitative behavior** of **discrete dynamical models**:

$$P_{n+1} = f(P_n)$$

**Equilibria:** First step of any **analysis** is finding **equilibria**, where there is no change in the **variable** or population from one iteration to the next.

Mathematically, this is solving the **algebraic equation**

$$P_e = f(P_e).$$

Graphically, this is equivalent to  $f(P_n)$  crossing the **identity map**,  $P_{n+1} = P_n$ .

**Note:** Any **closed population model**, meaning no migration into or out from another population source, must have an **equilibrium** of  $P_e = 0$ .

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## U. S. Population Models

The best fitting **discrete logistic population** model for the U. S. satisfies:

$$P_{n+1} = P_n + 0.2245P_n \left(1 - \frac{P_n}{451.7}\right),$$

while the best fitting **Beverton-Holt population** model for the U. S. satisfies:

$$P_{n+1} = \frac{1.23065P_n}{1 + \frac{P_n}{2110.5}}.$$

These are **closed discrete autonomous population** models, so have one equilibrium at  $P_e = 0$ .

We show there is another equilibrium, often called the **carrying capacity equilibrium**, where the population of these models eventually approach with sufficient iterations.

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## Logistic Population Model

Consider the *logistic population model*:

$$P_{n+1} = P_n + rP_n \left(1 - \frac{P_n}{M}\right).$$

The *equilibria* are found algebraically by solving:

$$P_e = P_e + rP_e \left(1 - \frac{P_e}{M}\right) \quad \text{or} \quad rP_e \left(1 - \frac{P_e}{M}\right) = 0.$$

Solving this (factored) quadratic gives the *extinction equilibrium*,  $P_e = 0$  and the *carrying capacity equilibrium*,  $P_e = M$ .

The best fitting *logistic population model* from the census data shows a *carrying capacity equilibrium*,  $P_e = 451.7$ .

Thus, this model predicts that the U. S. population will level off at 451.7M. (The current population from the Census Bureau website is approximately 326M.)



## Beverton-Holt Model

Consider the *Beverton-Holt model*:

$$P_{n+1} = \frac{aP_n}{1 + \frac{P_n}{b}}.$$

The *equilibria* are found algebraically by solving:

$$P_e = \frac{aP_e}{1 + \frac{P_e}{b}} \quad \text{or} \quad P_e \left(\frac{P_e}{b} + 1 - a\right) = 0.$$

Solving this (factored) quadratic gives the *extinction equilibrium*,  $P_e = 0$  and the *carrying capacity equilibrium*,  $P_e = b(a - 1)$ .

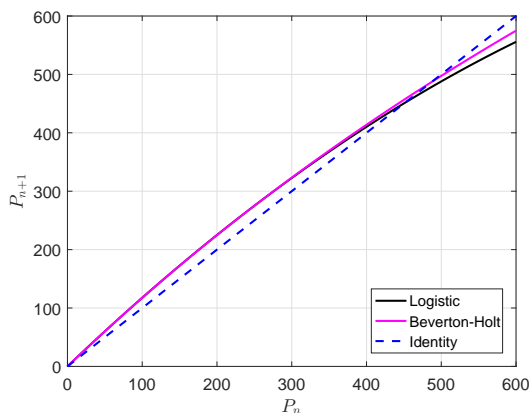
The best fitting *Beverton-Holt model* from the census data shows a *carrying capacity equilibrium*,  $P_e = 2110.5(0.23065) = 486.8$ .

Thus, this model predicts that the U. S. population will level off at 486.8M, similar to the *logistic population model*



## Graphing Updating Functions

Intersections of the *updating functions* for the *logistic* and *Beverton-Holt population models* with the *identity map* give the *equilibria* for the models.



## Stability of Equilibria

For very low populations, the population has plenty of resources and should grow according to the Malthusian growth model (exponentially).

It follows that often the *extinction equilibrium* has populations growing away from this equilibrium, so it is *unstable*.

On the other hand, populations tend to approach the *stable carrying capacity equilibrium*.

We want to develop *analytic tools* for determining the *stability* of the *equilibria*.

The primary analytic tool is taking the *derivative* at the *equilibria*.





## Stability of Logistic Population Model

The *updating function* of the best fitting *logistic population model* for the U. S. satisfies:

$$P_{n+1} = f(P_n) = P_n + 0.2245P_n \left(1 - \frac{P_n}{451.7}\right),$$

which has a derivative that satisfies:

$$f'(P) = 1.2245 - \frac{0.4490P}{451.7}.$$

At the *extinction equilibrium*,  $P_e = 0$ , we have

$$f'(0) = 1.2245 > 1,$$

which we'll see shows this equilibrium is *unstable*.

At the *carrying capacity equilibrium*,  $P_e = 451.7$ , we have

$$f'(451.7) = 0.7755 < 1,$$

which we'll see shows this equilibrium is *stable*.



## Stability of Beverton-Holt Model

The *updating function* of the best fitting *Beverton-Holt model* for the U. S. satisfies:

$$P_{n+1} = b(P_n) = \frac{1.23065P_n}{1 + \frac{P_n}{2110.5}},$$

which has a derivative that satisfies:

$$b'(P) = \frac{1.23065}{\left(1 + \frac{P}{2110.5}\right)^2}.$$

At the *extinction equilibrium*,  $P_e = 0$ , we have

$$b'(0) = 1.23065 > 1,$$

which we'll see shows this equilibrium is *unstable*.

At the *carrying capacity equilibrium*,  $P_e = 486.8$ , we have

$$b'(486.8) = 0.81257 < 1,$$

which we'll see shows this equilibrium is *stable*.

