

# Math 636 - Mathematical Modeling

## Bifurcation Analysis 1D ODEs

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## Introduction

### Introduction

- Our studies of the *competition of two species* and *predator-prey models* showed the importance of parameters in systems of ODEs.
- The parameters determined some of the observed behaviors.
- Often these parameters change for some reason or are governed by slow time dynamics.
- We return to examining *one-dimensional ODEs* and study how behaviors change as a parameter varies.
- The models depend on a parameter,  $\lambda$ , so

$$\frac{dy}{dt} = f(t, y, \lambda),$$

which can have varying behavior.



## Outline

- 1 Population Model with Constant Harvesting
  - Saddle Node Bifurcation
- 2 Population Models with Proportional Harvesting
  - Transcritical Bifurcation
- 3 Fluxgate Magnetometer
  - Pitchfork Bifurcation
- 4 Modeling Spruce Budworm
  - Ludwig, Jones, and Hollings Model
  - Dimensionless Variables
  - Bifurcation Analysis



## Population Models

### Population Models

With abundant resources, a population will grow exponentially, satisfying the Malthusian growth law:

$$\frac{dP}{dt} = rP,$$

where  $r$  is the growth rate.

With limited resources, the growth rate slows, and the next approximation to population growth satisfies the logistic growth equation given by:

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{M}\right),$$

where  $M$  is the carrying capacity.

Most populations tend to follow the classic *S-shaped curves*, which are the solutions of the *logistic growth model*.



## Population Models with Harvesting

There are a number of methods that humans use to harvest various animals or fish.

**Constant Harvesting:** One simple method is to set a limit for harvesting.

Assume the animals grow according to the *logistic growth model*, which are harvested at a constant rate,  $h$ :

$$\frac{dP}{dt} = rP \left( 1 - \frac{P}{M} \right) - h = F(P, h), \quad P(0) = p_0.$$

We want to study how harvesting at different levels affects the population.



A *population model with constant harvesting* satisfies:

$$\frac{dP}{dt} = rP \left( 1 - \frac{P}{M} \right) - h = F(P, h), \quad P(0) = p_0,$$

The analytic solution gives little insight into the model behavior, so *geometric techniques* are employed.

- Earlier efforts showed that *1D-phase portraits* are effective tools for finding behavior of *one-dimensional ODEs*.
- The technique for *qualitative analysis* of autonomous models only requires graphing the function,  $F(P, h)$ .
  - When  $F(P, h)$  crosses the  $P$ -axis, we find *equilibria*.
  - For  $F(P, h) > 0$ , the solutions are increasing, while for  $F(P, h) < 0$ , the solutions are decreasing.
  - This *geometric interpretation* allows easy visualization of the stability of the *equilibria*.



## Population Model with Constant Harvesting

The *logistic growth model* with harvesting is:

$$\frac{dP}{dt} = rP \left( 1 - \frac{P}{M} \right) - h = F(P, h), \quad P(0) = p_0.$$

If this ODE is inserted this into **Maple** to solve, then one obtains an explicit solution, which is very messy:

$$P(t) = \frac{1}{2r} \left( rM + K \tanh \left( \frac{K}{2M} \left( t + \frac{M}{K} \ln \left( \frac{2rp_0 + K - rM}{K + rM - 2rp_0} \right) \right) \right) \right),$$

where  $K = \sqrt{r^2 M^2 - 4hrM}$ .

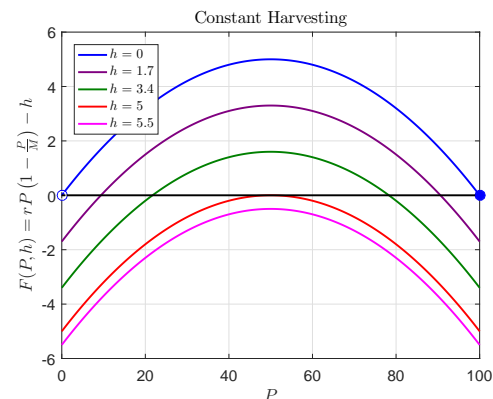
Clearly, this solution is difficult to analyze, so we turn to a geometric analysis of the model.



**Example:** A *population model with constant harvesting* satisfies:

$$\frac{dP}{dt} = 0.2P \left( 1 - \frac{P}{100} \right) - h = F(P, h).$$

Below is a graph of  $F(P, h)$  with different *harvesting levels*.



Example with Constant Harvesting

2

**Example (cont):** The *population model* given by:

$$\frac{dP}{dt} = 0.2P \left( 1 - \frac{P}{100} \right) - h = F(P, h), \quad P(0) = p_0,$$

with  $h = 0$  (no harvesting) is the *logistic growth model*.

All solutions with  $0 < p_0 < 100$  grow away from the *extinction equilibrium* or  $P_e = 0$ , so this *equilibrium* is *unstable*.

All solutions with  $p_0 > 0$  grow toward the *carrying capacity*,  $P_e = M$ , so this *equilibrium* is *stable*.

The graph shows a *parabola* pointing down:

- The *vertex*,  $(P_v, F(P_v)) = (50, 5)$ , is the maximum growth of the population.
- The  $P$ -intercepts are the *equilibria*.
- The *constant harvesting*,  $h$ , shifts the vertex downward with increasing  $h$ .



Example with Constant Harvesting

3

**Example (cont):** The *equilibria* for the *population model* with constant harvesting satisfies:

$$0.2P_e \left( 1 - \frac{P_e}{100} \right) - h = 0.$$

We interpret the dynamical system from a modeling perspective.

Suppose that  $0 < h < 5$ .

- There remain two *equilibria* with an *unstable equilibrium* satisfying  $0 < P_{1e}(h) < 50$  and a *stable equilibrium* satisfying  $P_{2e}(h) > 50$ .
- Once again the stable equilibrium,  $P_{2e}(h)$ , will be the *carrying capacity of this population*.
- If the *constant harvesting*,  $h$ , is phased in where the population before harvesting begins is near *carrying capacity*, then we expect a measured decline of the population with increasing  $h$  with the new equilibrium,  $P_{2e}(h)$ .



Example with Constant Harvesting

4

**Example (cont):** Continuing with the case  $0 < h < 5$ .

- The lower *unstable equilibrium* satisfies  $0 < P_{1e}(h) < 50$ , so any initial condition  $p_0 > P_{1e}(h)$  will have the population increase to the *carrying capacity*,  $P_{2e}(h)$ .
- However, any initial population with  $0 < p_0 < P_{1e}(h)$  has the population declining.
- Since there is no lower equilibrium, then the population must decline with  $P(t) \rightarrow -\infty$ , which is obvious nonsense.
- This suggests that the model needs serious revision to account for lower populations.
- It should be clear that one cannot keep harvesting a fixed limit,  $h$ , when there are low populations.



Example with Constant Harvesting

5

**Example (cont):** Considering the case  $h = 5$ .

- When  $h = 5$ , the *two equilibria* coalesce, so  $P_{1e}(5) = P_{2e}(5) = 50$ .
- This results in a *degenerate equilibrium*, which is *half stable*.
- Any initial condition with  $p_0 > 50$  will see the population move to the single equilibrium, so  $P(t) \rightarrow P_{2e}(5) = 50$ .
- Any initial condition with  $p_0 < 50$  will see the population move away from  $P_{1e}(5)$ , so  $P(t) \rightarrow -\infty$ , which is obvious nonsense.
- This value of the parameter  $h = 5$  is called a *saddle node bifurcation*.
- The *ODE model* has a distinct change of behavior from *two equilibria* to *no equilibria* as the parameter  $h$  changes from below 5 to above.

When  $h > 5$ , then the harvesting is too intense and the animal population cannot survive.

However, the model is unrealistic in its harvesting assumption, as it shows  $P(t) \rightarrow -\infty$ .



**Saddle Node Bifurcation:** The classic form of the *saddle node bifurcation* is the 1D ODE:

$$\dot{y} = \mu - y^2,$$

where  $\mu$  is the bifurcation parameter.

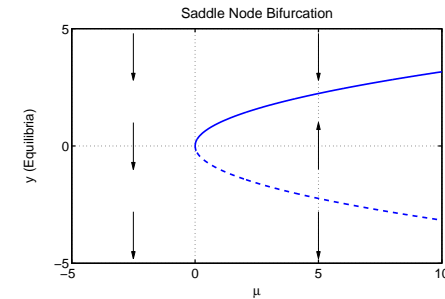
- As the parameter,  $\mu$ , varies, the behavior of the system changes or *bifurcates* at  $\mu = 0$ .
- At the *bifurcation point* equilibria either appear or disappear, depending on the parameter,  $\mu$ .
- This bifurcation is often called a *blue sky bifurcation* because the appearance of equilibria appear out of nowhere.



**Saddle Node Bifurcation:** The *bifurcation diagram* for the equation:

$$\dot{y} = \mu - y^2,$$

the horizontal axis as the parameter,  $\mu$ , and the vertical axis shows the value of the equilibria for a particular value of  $\mu$ .



The solid line indicates a *stable equilibrium*, and the dotted line indicates an *unstable equilibrium*.



### Population Model with Proportional Harvesting

Often one puts in a constant effort into harvesting, such as netting fish, which yields *harvesting proportional to the population*.

The *logistic growth model* with proportional harvesting is:

$$\frac{dP}{dt} = rP \left( 1 - \frac{P}{M} \right) - hP = G(P, h), \quad P(0) = p_0.$$

If this ODE is inserted this into **Maple** to solve, then one obtains an explicit solution:

$$P(t) = \frac{p_0 M (r - h)}{p_0 r + (M(r - h) - p_0 r) e^{-(r-h)t}}.$$

Assuming  $r > h$ , then one readily shows that

$$\lim_{t \rightarrow \infty} P(t) = \frac{M(r - h)}{r}.$$



A *population model with constant harvesting* satisfies:

$$\frac{dP}{dt} = rP \left( 1 - \frac{P}{M} \right) - hP = G(P, h), \quad P(0) = p_0,$$

The analytic solution gives some insight into the model behavior, but *geometric techniques* provide more intuition.

- Equilibrium analysis*, where  $G(P_e, h) = 0$ , gives

$$P_e = 0 \quad \text{or} \quad P_e = \frac{M(r - h)}{r}.$$

- $P_e = 0$  is the *unstable, extinction equilibrium*, while  $P_e = \frac{M(r-h)}{r}$  is the *stable, carrying capacity equilibrium*, provided  $r > h$ .
- The *qualitative analysis* uses graphing the function,  $G(P, h)$ , where for  $G(P, h) > 0$ , the solutions are increasing, while for  $G(P, h) < 0$ , the solutions are decreasing.
- This *geometric interpretation* allows easy visualization of the stability of the *equilibria*.



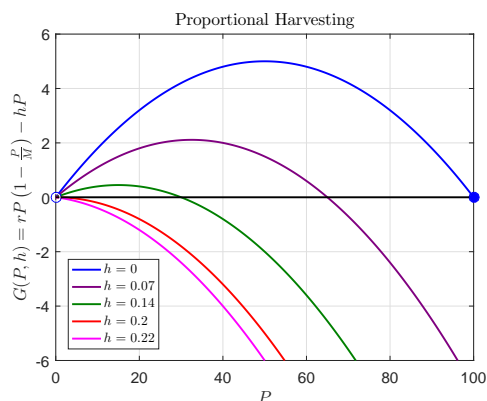
## Example with Proportional Harvesting

1

**Example:** A *population model with proportional harvesting* satisfies:

$$\frac{dP}{dt} = 0.2P \left( 1 - \frac{P}{100} \right) - hP = G(P, h).$$

Below is a graph of  $G(P, h)$  with different *harvesting levels*.



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## Example with Proportional Harvesting

2

**Example (cont):** The *proportional harvesting model* given by:

$$\frac{dP}{dt} = 0.2P \left( 1 - \frac{P}{100} \right) - hP = G(P, h),$$

has the *equilibria*, where  $G(P_e, h) = 0$ ,

$$P_e = 0 \quad \text{or} \quad P_e = 500(0.2 - h).$$

The graph shows a *parabola* pointing down:

- If  $0 \leq h < 0.2$ , then there is a positive *carrying capacity equilibrium*,  $P_e = 500(0.2 - h)$ .
- All solutions with  $p_0 > 0$  grow toward this *equilibrium*.
- If  $h > 0.2$ , then there is a negative *equilibrium* and the *extinction equilibrium*,  $P_e = 0$  becomes a *stable equilibrium*.
- This scenario has harvesting exceeding reproduction.
- At  $h = 0.2$ ,  $P_e = 0$  is the only equilibrium (and it is *half-stable*) leading to a *transcritical bifurcation* at this  $h$ .

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## Transcritical Bifurcation

1

**Transcritical Bifurcation:** The classic form of the *transcritical bifurcation* is the **1D ODE**:

$$\dot{y} = \mu y - y^2,$$

where  $\mu$  is the bifurcation parameter.

- Note that  $y_e = 0$  is always an *equilibrium*.
- The other *equilibrium* occurs at  $y_e = \mu$ .
- Geometrically, the equation  $f(y, \mu) = \mu y - y^2$  is a parabola pointing down, intersecting the  $\mu$ -axis at 0 and  $\mu$ .
- Thus, from before the larger equilibrium is *stable*, while the smaller equilibrium is *unstable*.
- For  $\mu < 0$ ,  $y_e = 0$  is *stable*, while  $y_e = \mu$  is *unstable*. The reverse holds for  $\mu > 0$  ( $y_e = 0$  (*unstable*) and  $y_e = \mu$  (*stable*)).
- At  $\mu = 0$ ,  $y_e = 0$  changes stability leading to what is called a *transcritical bifurcation*.

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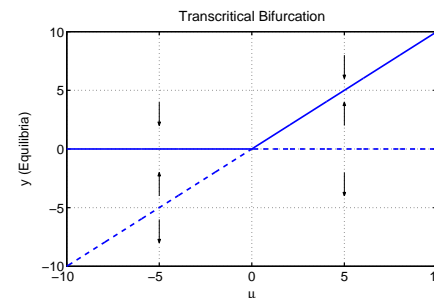
## Transcritical Bifurcation

2

**Transcritical Bifurcation:** The *transcritical diagram* for the equation:

$$\dot{y} = \mu y - y^2,$$

the horizontal axis as the parameter,  $\mu$ , and the vertical axis shows the value of the equilibria for a particular value of  $\mu$ .



The solid line indicates a *stable equilibrium*, and the dashed line indicates an *unstable equilibrium*.

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## Fluxgate Magnetometer

**Fluxgate Magnetometer:** A number of models that have a switching mechanism use the hyperbolic tangent function.

A one dimensional model that has been used for a *fluxgate magnetometer* (and *nerve impulse models*) satisfies the equation:

$$\dot{y} = -y + \tanh(cy) = H(y, c),$$

where  $c$  is a temperature dependent parameter that controls the behavior of the potential function from which this ODE is derived.

The *equilibria* are found by solving:

$$y_e = \tanh(cy_e).$$

- This always has the solution  $y_e = 0$ .
- If  $c < 1$ , then  $y_e = 0$  is the only solution.
- If  $c > 1$ , then there are always three equilibria.



## Fluxgate Magnetometer

The *fluxgate magnetometer model* satisfies:

$$\dot{y} = -y + \tanh(cy) = H(y, c),$$

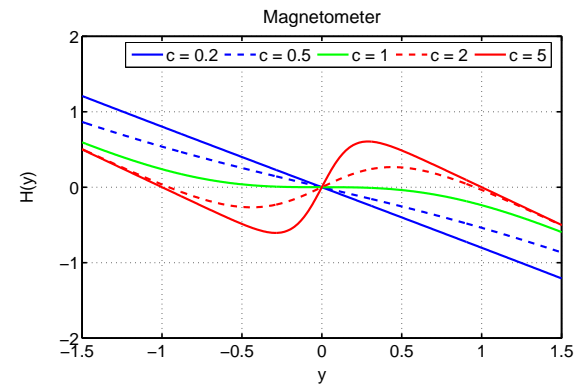
where again we use *geometric analysis* to study the *qualitative theory* of this ODE.

- The *equilibria* occur where  $H(y, c)$  crosses the  $y$ -axis.
- If  $H(y, c) > 0$ , then the solution,  $y(t)$  is increasing in time, while if  $H(y, c) < 0$ , then the solution,  $y(t)$  is decreasing.
- An *equilibrium* is *stable* if the solution is **increasing to the left and decreasing to the right**.
- Similarly, an *equilibrium* is *unstable* if the solution is **increasing to the left and decreasing to the right** of  $y_e$ .



## Fluxgate Magnetometer

Below is a graph of  $H(y, c)$ . The equilibria ( $H(y_e, c) = 0$ ) are on the  $y$ -axis with solutions increasing for  $H(y) > 0$  and decreasing for  $H(y) < 0$ .



This graph show  $y_e = 0$  losing stability at  $c = 1$  with two new *equilibria* appearing.



## Pitchfork Bifurcation

**Pitchfork Bifurcation:** The classic form of the *pitchfork bifurcation* is the **1D ODE**:

$$\dot{y} = \mu y - y^3,$$

where  $\mu$  is the bifurcation parameter.

- Note that  $y_e = 0$  is always an *equilibrium*.
- The other *equilibria* occur at  $y_e = \pm\sqrt{\mu}$ , provided  $\mu > 0$ .
- Geometrically, the equation  $f(y, \mu) = \mu y - y^3$  is a cubic polynomial intersecting the  $\mu$ -axis at 0 and  $\pm\sqrt{\mu}$  if  $\mu > 0$ .
- For  $\mu < 0$ ,  $y_e = 0$  is *stable*, as  $y(t)$  increases for  $y < 0$  and decreases for  $y > 0$  (cubic is monotonic).
- For  $\mu > 0$ ,  $y_e = 0$  becomes *unstable* and the two new *stable equilibria* ( $y_e = \pm\sqrt{\mu}$ ) arise.
- At  $\mu = 0$  we have what is called a *pitchfork bifurcation*.





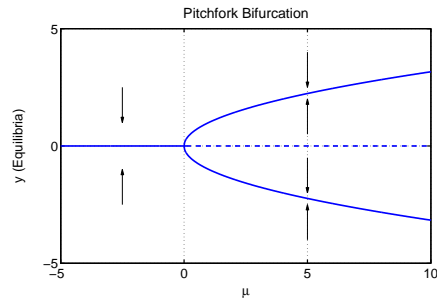
## Pitchfork Bifurcation

2

**Pitchfork Bifurcation:** The *pitchfork diagram* for the equation:

$$\dot{y} = \mu y - y^3,$$

the horizontal axis as the parameter,  $\mu$ , and the vertical axis shows the value of the equilibria for a particular value of  $\mu$ .



The solid line indicates a *stable equilibrium*, and the dashed line indicates an *unstable equilibrium*.



## Spruce Budworm

2

### Spruce Budworm:

- There are about a dozen species of the genus *Choristoneura* that range through the boreal forests of U. S. and Canada.
- The species, *Choristoneura fumiferana* or eastern spruce budworm, is one of the most destructive native insect species in northern spruce and fir forests, particularly affecting balsam fir trees in Eastern U. S. and Canada.
- There are periodic outbreaks of this pest with the first recorded outbreak occurring in Maine around 1807.
- Another another outbreak in 1878.
- From tree ring studies, scientists have determined that outbreaks of the spruce budworm have occurred every 40-60 years, since at least the 16<sup>th</sup> century.
- Eastern Canada has seen three major outbreaks between 1910-1920, 1940-1950, and 1970-1980.
- Every 30-60 years there is a serious outbreak of the pest that results in devastation of a forest, particularly the balsam firs in about four years.



## Spruce Budworm

1

Most species of **spruce budworm** are serious pests that destroy coniferous forests.



Juvenile



Adult



## Spruce Budworm Modeling

1

### Spruce Budworm Modeling:

- There are mathematical and ecological models for these periodic outbreaks.
- One theory considers spatially synchronized population oscillations caused by delayed density-dependent feedback that has high amplitude oscillations from entrainment.
- Another theory uses cusp-catastrophe theory with populations jumping between endemic and epidemic levels.
- This study combines a one-dimensional nonlinear ODE with a slow varying parameter.
- The spruce budworm is always present, but it is usually controlled by birds predated on the insect.
- Our model examines the rapid dynamics of the predator-prey interactions of birds (a smart predator) and its prey the spruce budworm.
- The growth of the spruce is on a slow scale, and it is the primary food of the spruce budworm.



**Spruce Budworm Modeling:** Below are some facts and assumptions in the model.

- The model examines the complicated interaction of the spruce budworm population with its host, the coniferous trees, and its primary predator, birds.
- There are significant differences in time scales between these particular species.
  - Spruce budworms reproduce very rapidly with the ability to increase 5-fold in a single year.
  - The balsam fir can replace its foliage in about 7 years and can live for about 100 years.
- Assume that the birds are smart opportunistic predators with plenty of different sources of food.
- Assume birds maintain a relatively constant population.
- Assume that the needles of the balsam fir are the primary source of food for the spruce budworms, and these leaves provide both nourishment and hiding for the spruce budworms.



This *spruce budworm model* is based on work of Ludwig, Jones, and Hollings (1978), which became important through a *Nature* article of Robert May (1977).

- It is a simple ODE that tracks the population of the spruce budworm.
  - Parameters account for their interactions with their food source, the balsam firs.
  - And their predators, the birds.
- The bird population is assumed to be constant because of limitations on territorial factors and not food sources.
- Its predatory behavior is considered a learned behavior, which satisfies a Holling's Type III interaction.
- The foliage coverage of the trees is a slow dynamic compared to the spruce budworm generation time.
  - It is a slow dynamic in the model.
  - Often referred to as a slow moving parameter.



The *Ludwig, Jones, and Holling model* has the following form:

$$\frac{dB}{dt} = r_B B \left( 1 - \frac{B}{K_B} \right) - g(B).$$

- $r_B$  is the intrinsic growth rate of the spruce budworm.
- $K_B$  is the carrying capacity, which depends on the availability of the foliage of the balsam firs.
- The rate of predation,  $g(B)$ , is from avian predators, which associate reward with prey, a learned behavior.
  - They focus on their best sources of prey, which allows a low density prey to escape notice.
  - The functional form of this type of predation follows the *Holling's Type III* S-shaped response given by:

$$g(B) = \beta \frac{B^2}{\alpha^2 + B^2}.$$

- $\beta$  represents the saturation level of the predator (constant population only eats so much prey).
- $\alpha$  determines the densities of spruce budworm that cause avian



The *spruce budworm model* satisfies:

$$\frac{dB}{dt} = r_B B \left( 1 - \frac{B}{K_B} \right) - \beta \frac{B^2}{\alpha^2 + B^2}.$$

- This model above has 4 parameters.
- We analyze the behavior of this equation when the slow varying parameters are held constant.
- It is convenient to scale out two of the parameters and analyze the model depending on only two scaled parameters.
- The two parameters are scaled out by changing the population scale and the time scale.





## Scaling the Variables

1

Scale the population and time by:

$$\mu = sB \quad \text{and} \quad \tau = qt.$$

A scaled differential equation is written:

$$\begin{aligned} \frac{d\mu}{d\tau} &= \frac{s}{q} \frac{dB}{dt} = \frac{s}{q} \left( r_B B \left( 1 - \frac{B}{K_B} \right) - \beta \frac{B^2}{\alpha^2 + B^2} \right) \\ &= \frac{s}{q} \left( r_B \frac{\mu}{s} \left( 1 - \frac{\mu}{sK_B} \right) - \beta \frac{(\mu/s)^2}{\alpha^2 + (\mu/s)^2} \right) \end{aligned}$$

Let  $s = 1/\alpha$ , then

$$\frac{d\mu}{d\tau} = \frac{r_B}{q} \mu \left( 1 - \frac{\alpha\mu}{K_B} \right) - \frac{\beta}{\alpha q} \left( \frac{\mu^2}{1 + \mu^2} \right).$$

Take  $q = \beta/\alpha$ , then

$$\frac{d\mu}{d\tau} = \frac{\alpha r_B}{\beta} \mu \left( 1 - \frac{\alpha\mu}{K_B} \right) - \frac{\mu^2}{1 + \mu^2}.$$



## Scaling the Variables

3

The scaled model satisfies:

$$\frac{d\mu}{d\tau} = R\mu \left( 1 - \frac{\mu}{Q} \right) - \frac{\mu^2}{1 + \mu^2}.$$

The non-extinction *equilibria* satisfy the equation:

$$f(\mu) = R \left( 1 - \frac{\mu}{Q} \right) = \frac{\mu}{1 + \mu^2} = g(\mu).$$

- We give an ecological perspective of this equation.
- The left hand side of the equation,  $f(\mu)$ , is the *per capita* growth rate of the scaled population variable,  $\mu$ , (with respect to the scaled time,  $\tau = \beta t/\alpha$ ).
- The scaled equation on the right hand side,  $g(\mu)$ , is the *per capita* death rate of the spruce budworms due to avian predation.



## Scaling the Variables

2

If we define the new scaled parameters,

$$R = \frac{\alpha r_B}{\beta} \quad \text{and} \quad Q = \frac{K_B}{\alpha},$$

then the scaled model is given by:

$$\frac{d\mu}{d\tau} = R\mu \left( 1 - \frac{\mu}{Q} \right) - \frac{\mu^2}{1 + \mu^2}.$$

A simple linearization near the equilibrium,  $\mu_e = 0$ , gives

$$\frac{d\mu}{d\tau} = R\mu,$$

which is Malthusian growth and gives the natural result that at low densities the spruce budworm population grows exponentially.



## Scaling the Variables

3

The non-extinction *equilibria* satisfy the equation:

$$f(\mu) = R \left( 1 - \frac{\mu}{Q} \right) = \frac{\mu}{1 + \mu^2} = g(\mu).$$

- Points of intersection of the curves formed by the two sides of the equation above produce non-zero equilibria for the differential equation for  $\mu$ .
- The equation on the left hand side is a straight line with intercepts at  $\mu = Q$  and a  $y$ -intercept at  $R$ .
- The right hand side intersects the origin and has a horizontal asymptote of  $y = 0$ .
- The function,  $g(\mu)$ , has a maximum at  $(1, \frac{1}{2})$ .



**Bifurcation Analysis:** The scaled differential equation for the *spruce budworm population model* satisfies:

$$\frac{d\mu}{d\tau} = R\mu \left(1 - \frac{\mu}{Q}\right) - \frac{\mu^2}{1 + \mu^2}.$$

- The parameter  $Q$  is approximately constant.
  - The *carrying capacity*  $K_B$  scales with the parameter  $\alpha$ .
  - $\alpha$  measures the densities for predators to change prey sources.
- The parameter  $R$  is a more complex combination of parameters.
  - It slowly increases at low densities.
  - It decreases when there is an outbreak of the spruce budworm that causes massive defoliation of the forests.

**Equilibria:** Clearly, there is the *trivial* or *extinction equilibrium*,  $\mu_e = 0$ .

*Linear analysis* shows that this equilibrium is *unstable*.

When  $\frac{d\mu}{d\tau} = 0$ , the other equilibria satisfy:

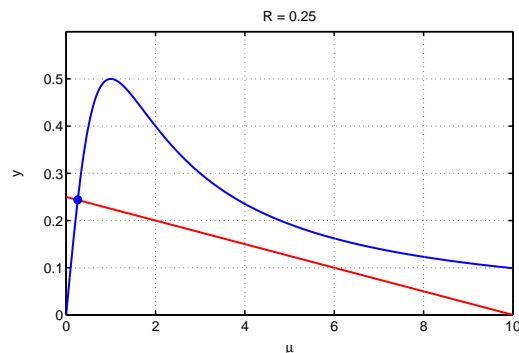
$$f(\mu) = R \left(1 - \frac{\mu}{Q}\right) = \frac{\mu}{1 + \mu^2} = g(\mu),$$

which has **one to three** solutions.

The solutions to this equation are seen by graphing the  $f(\mu)$  and  $g(\mu)$  for different values of  $R$  ( $Q$  fixed).

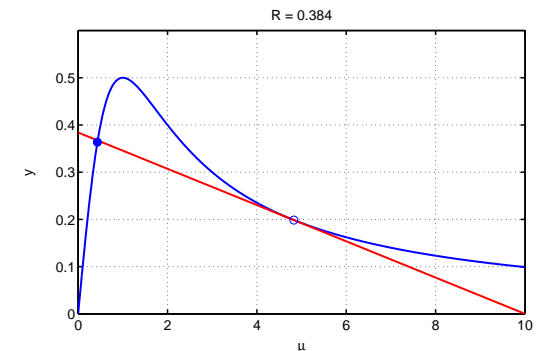
Begin with a small value of  $R$ , say  $R = 0.25$ . ( $Q = 10$ )

- The graph shows there is only one additional equilibrium at  $\mu_e = 0.260$ .
- This equilibrium is *stable*.
- Biologically, this scaled population represents an endemic population of spruce budworms in a healthy forest.



As the budworm population grows and  $R$  increases.

- The line,  $f(\mu)$ , becomes tangent to the curve,  $g(\mu)$ , at  $R = 0.38397$ .
- This is a *saddle node (blue sky)* bifurcation with a new equilibrium appearing at  $\mu_e = 4.828$ .

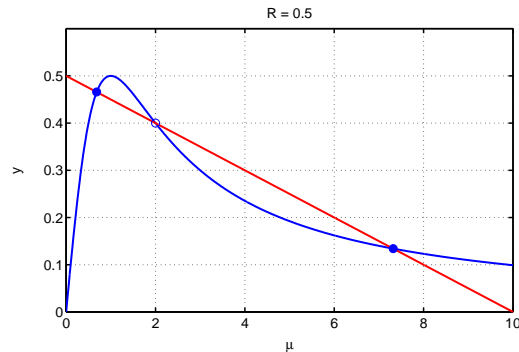


## Bifurcation Analysis

5

As  $R$  increases, two new equilibria appear with the lower one being *unstable* and the upper one being *stable*.

- The solution of the model remains trapped near the lower stable equilibrium.
- The spruce budworm population remains endemic, and the forest remains healthy and can grow with the bird population keeping the budworms sufficiently in check.



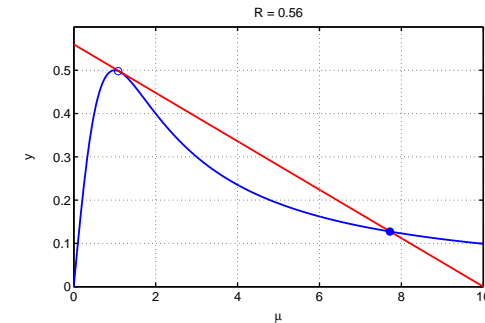
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## Bifurcation Analysis

6

As  $R$  increases to  $R = 0.5595$ , another *saddle node (blue sky)* bifurcation in the reverse direction results in the lower two equilibrium points vanishing.

- This results in only having a single *stable equilibrium* with a high value.
- Now there is an outbreak of the spruce budworms in large numbers as they saturate the ability of the predators to control their population.



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## Bifurcation Analysis

7

After  $R$  exceeds 0.5595, the population goes to a *carrying capacity*.

- This causes massive defoliation of the forests.
- The change in the model results in the slow dynamics of the parameter  $R$  with  $R$  beginning to decrease.
- As defoliation continues, the solution passes through the saddle node bifurcation at  $R = 0.5595$ .
- Now the solution remains at the larger equilibrium.
- The health of the forest continues to decline.
- This continues until the budworms so severely damage the forest that their population collapses.
- At  $R = 0.38397$ , a second *saddle node bifurcation* occurs.
- The only remaining equilibrium is the *stable endemic equilibrium*.
- From this point, the cycle between the growth of the forest and the outbreak of spruce budworms continues in a cyclical manner.

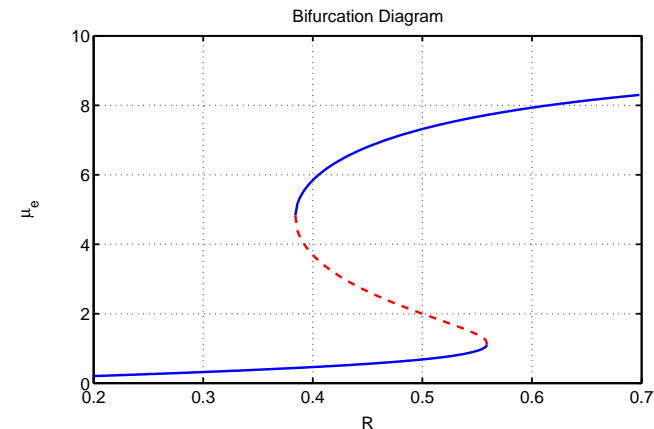
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## Bifurcation Analysis

8

This *dynamical model* has fast dynamics for the population of the spruce budworm,  $\mu$ , and includes a slowly varying parameter,  $R$ .

This type of model fits something called *Catastrophe theory*.



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