Math 636 - Mathematical Modeling

Allometric Modeling and Dimensionless Systems

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Outline

Kleiber's Relationship

Metabolism and Size

- Kleiber asks, "Does a horse produce more heat per day than a rat...?"[1]
- Obviously, **YES**
- "Does a horse produce more heat per day per kilogram of body weight than a rat?"
- Clearly, **NO**
- Animals benefit metabolically by increasing size

[1] Max Kleiber (1947), "Body size and metabolic rate," *Physiological Reviews*, **24**, 511-541

Metabolism/Weight for Animals

Table of Metabolism (kcal) and Weight (kg) for Various Animals

Animal	Weight	Metabolism	Animal	Weight	Metabolism
Mouse	0.021	3.6	Dog	24.8	875
Rat	0.282	28.1	Dog	23.6	872
Guinea pig	0.41	35.1	Goat	36	800
Rabbit	2.98	167	Chimpanzee	38	1090
Rabbit	1.52	83	Sheep	46.4	1254
Rabbit	2.46	119	Sheep	46.8	1330
Rabbit	3.57	154	Woman	57.2	1368
Rabbit	4.33	191	Woman	54.8	1224
Rabbit	5.33	233	Woman	57.9	1320
Cat	3	152	Cow	300	4221
Macque	4.2	207	Cow	435	8166
Dog	6.6	288	Heifer	482	7754
Dog	14.1	534	Cow	600	7877

Modeling Data

Modeling the Data

- The data are clearly not linear
- There are general methods for finding the least squares best fit to nonlinear data
- These techniques are very complicated and often difficult to implement
- **Power Law** or **Allometric Models** are easier

Allometric Models or Power Law Model

Allometric Models

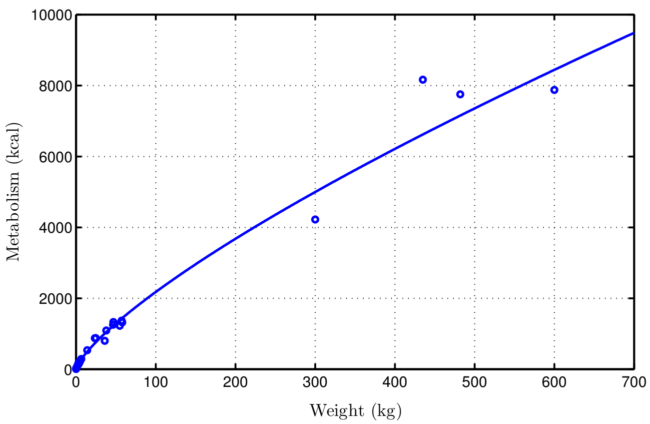
- Allometric models assume a relationship between two sets of data, x and y, that satisfy a power law of the form

$$y = Ax^r$$

- A and r are parameters that are chosen to best fit the data in some sense
- This model assumes that when $x = 0$, then $y = 0$
- The method fits a straight line to the logarithms of the data

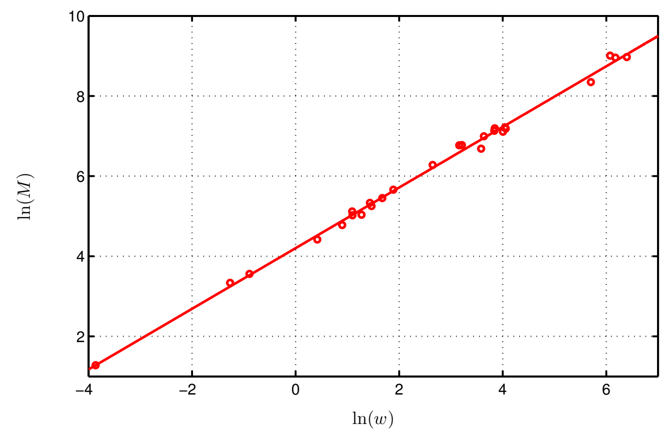
Allometric Model of Kleiber's Law

Graph of the Metabolic and Weight data



Allometric Model of Kleiber's Law

If w is the weight (kg) and M is the metabolic rate (kcal), then below is the graph of $\ln(M)$ vs. $\ln(w)$



Allometric Model of Kleiber's Law

Allometric Model of Kleiber's Law

- The best slope is $r = 0.7565$
- The best intercept is $\ln(A) = 4.202$ with $A = 66.82$
- This gives the best fit power law for this model as

$$M = 66.82w^{0.7565}$$

- The minimum least squares for the log of the data gives $J(A,r) = 3.81 \times 10^6$
- Nonlinear least squares best fit model (with a better fit (SSE) $J(A,r) = 3.64 \times 10^6$) satisfies

$$M = 63.86w^{0.7685}$$

Kleiber's Law

Allometric Model of Kleiber's Law

$$M = 66.82w^{0.7565}$$

- The graph of the power law provides a reasonable fit to the data
- The logarithm of the data closely lie on a straight line
- The coefficient $A = 66.82$ scales the variables
- The power $r = 0.7565$ often give physical insight to the behavior
  - If metabolism rate was proportional to mass, then $r = 1$
  - If metabolism relates to heat loss through skin, we expect $r = \frac{2}{3}$
  - **Why is $r = \frac{3}{4}$? This is Kleiber's Law.**

Allometric Model/Power Law

Allometric Model/Power Law

- When the logarithm of the data lie on a line, then a **Allometric Model** is appropriate
- Allometric Model can give insight into underlying mechanics of a problem
- Numerous examples satisfy allometric models
- **Excel** uses this logarithmic fit to data with a ***linear least squares*** with its ***Power Law fit*** under ***Trendline***

MatLab – Allometric Model/Power Law

MatLab: Below is a code for the logarithmic fit to data with a ***linear least squares***

```
function [k,a] = powerfit(xdata,ydata)
% Power law fit for model y = k*x^a
% Uses linear least squares fit to logarithms of data
Y = log(ydata);       % Logarithm of y-data
X = log(xdata);       % Logarithm of x-data
p = polyfit(X,Y,1);   % Linear fit to X and Y
a = p(1);             % Value of exponent
k = exp(p(2));        % Value of leading coefficient
end
```

MatLab – Allometric Model/Power Law

MatLab: Below is a code for ***nonlinear least squares*** fit to data

First define a ***sum of square errors*** function depending on the data and the parameters in the model

```
function J = sumsq_pow(p,xdata,ydata)
% Function to compute the least squares error for ...
    allometric model
model = p(1)*xdata.^p(2);  % Power law model ...
    using parameter p
error = model - ydata;      % Error between model ...
    and data
J = error*error';           % Computes sum of ...
    square error
end
```

Next use **MatLab's** nonlinear solver

```
p1=fminsearch(@sumsq_pow,p0,[],x,y)
```


Buckingham Pi Theorem

Theorem (Buckingham Pi Theorem)

Let q_1, q_2, q_3, ..., q_n be n dimensional variables that are physically relevant in a given problem and that are inter-related by an (unknown) dimensionally homogeneous set of equations. These can be expressed via a functional relationship of the form:

$$F(q_1, q_2, ...q_n) = 0 \qquad \text{or equivalently} \qquad q_1 = f(q_2, ...q_n).$$

If k is the number of fundamental dimensions required to describe the n variables, then there will be k primary variables and the remaining variables can be expressed as $(n-k)$ dimensionless and independent quantities or Pi groups, $\Pi_1, \Pi_2, ..., \Pi_{n-k}$. The functional relationship can tbe reduced to the much more compact form:

$$\Phi(\Pi_1, \Pi_2, , \Pi_{n-k}) = 0 \qquad \text{or equivalently} \qquad \Pi_1 = \Phi(\Pi_2, , \Pi_{n-k}).$$

Rayleigh's Method of Dimensional Analysis

Rayleigh's method of dimensional analysis

- Gather all the independent variables that are likely to influence the dependent variable.
- If R is a variable that depends upon independent variables $R_1, R_2, R_3, ..., R_n$, then the functional equation can be written as $R = F(R_1, R_2, R_3, ..., R_n)$.
- Write the above equation in the form $R = CR_1^a R_2^b R_3^c ... R_n^m$, where C is a dimensionless constant and $a, b, c, ..., m$ are arbitrary exponents.
- Express each of the quantities in the equation in some base units in which the solution is required.
- By using dimensional homogeneity, obtain a set of simultaneous equations involving the exponents $a, b, c, ..., m$.
- Solve these equations to obtain the value of exponents $a, b, c, ..., m$.
- Substitute the values of exponents in the main equation, and form the non-dimensional parameters by grouping the variables with like exponents.

Dimensional Analysis

Dimensional Analysis – Primary Units

There are a number of primary units:

Length	Mass	Time	Amount	Temperature	Electricity	Luminosity
L	M	T	N	Q	I	C

Example 1: Newton's Law of Force is given by

$$F = ma$$

This could be written

$$\frac{F}{ma} - 1 = 0,$$

which gives the ***dimensionless quantity***

$$\Pi = \frac{F}{ma}, \qquad \text{so} \qquad f(\Pi) = \Pi - 1.$$

Dimensional Analysis – Example 1

Example - Launching: Consider launching an object with critical quantities: $m =$ mass, $v =$ launch velocity, $h =$ maximum height, and $g =$ acceleration gravity

Choose:

$$[m] = M \qquad [v] = LT^{-1} \qquad [h] = L \qquad [g] = LT^{-2}$$

Create the ***dimensionless quantity***:

$$\Pi = m^a v^b h^c g^d$$

Analyze the exponents for quantities M, L, and T, so to be dimensionless

$$a = 0 \qquad b + c + d = 0 \qquad -b - 2d = 0.$$

Dimensional Analysis – Example 2

Example (cont): There are **4** coefficients a, b, c, and d for the **3** dimensional variables M, L, and T, leaving one free parameter.

With the one degree of freedom, we take $d = c$ and $c = 1$, then the coefficients become

$$a = 0 \qquad b = -2 \qquad c = 1 \qquad d = 1.$$

The dimensionless variable is

$$\Pi = \frac{hg}{v^2} \qquad f(\Pi) = f\left(\frac{hg}{v^2}\right) = 0.$$

It follows that

$$\frac{hg}{v^2} = k \qquad \text{or} \qquad h = \frac{kv^2}{g}.$$

Dimensional Analysis – Example 3

Example (cont): Since

$$h = \frac{kv^2}{g},$$

it follows that the ***height of a launch*** depends only on the quantity v^2/g.

- The ***height of a launch*** is independent of the ***mass***.
- The ***height of a launch*** varies as the square of the ***velocity***.
- The ***height of a launch*** is inversely proportional to the acceleration of ***gravity***.

It follows that doubling the launch velocity increases the height of the launch by a factor of 4.
On the moon with gravity, $\frac{g}{6}$, the height of the launch increases by a factor of 6.

Dimensional Analysis – Atomic Bomb 1

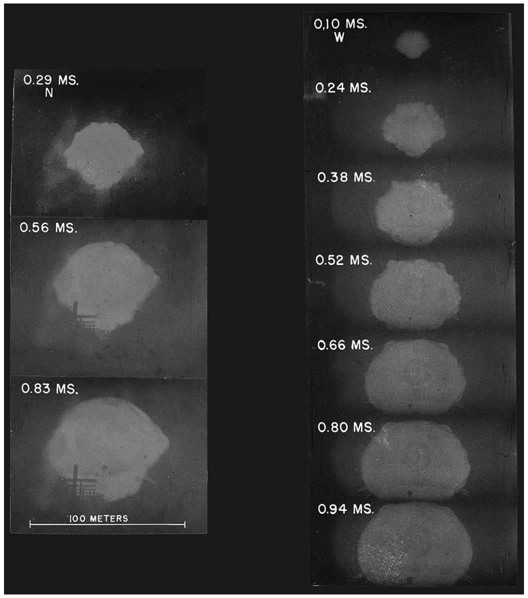
Example – Atomic Bomb: Sir Geoffrey Taylor F.R.S., The formation of a blast wave by a very intense explosion: II. The atomic explosion of 1945, *Proc. R. Soc. Lond.*, **A**, (1950)

This article used a movie of the Trinity test with dimensional analysis to estimate the power of the explosion

Pictures of the White Sands, NM test in 1945 showed the radius of explosion:

Time, t (sec)	Blast Radius, R (m)	Time, t (sec)	Blast Radius, R (m)
0.00038	25.4	0.0008	34.2
0.00052	28.8	0.00094	36.3
0.00066	31.9	0.00108	38.9

Dimensional Analysis – Atomic Bomb 2



Dimensional Analysis – Atomic Bomb 3

Atomic Bomb (cont): Assume that the radius, R, of Atomic blast depends only on time, t, ambient density, ρ, and Energy, E, of the explosion – we ignore other effects

From the Buckingham Pi Theorem, the ***dimensionless variable*** satisfies:

$$\Pi = R^a E^b t^c \rho^d,$$

where

$$[R] = L \qquad [E] = \frac{ML^2}{T^2} \qquad [t] = T \qquad [\rho] = \frac{M}{L^3},$$

so

$$\Pi = L^a \left(\frac{ML^2}{T^2}\right)^b T^c \left(\frac{M}{L^3}\right)^d.$$

Dimensional Analysis – Atomic Bomb 4

Atomic Bomb (cont): From before, the ***dimensionless variable*** satisfies:

$$\Pi = L^a \left(\frac{ML^2}{T^2}\right)^b T^c \left(\frac{M}{L^3}\right)^d.$$

From the coefficients above we have

$$\begin{aligned} a + 2b - 3d &= 0 \quad (L) \\ b + d &= 0 \quad (M) \\ -2b + c &= 0 \quad (T) \end{aligned}$$

There is one degree of freedom, so let $b = 1$, then

$$a = -5 \qquad b = 1 \qquad c = 2 \qquad d = -1.$$

Dimensional Analysis – Atomic Bomb 5

Atomic Bomb (cont): From the ***dimensionless variable***, we write

$$\Pi = R^{-5} E t^2 \rho^{-1} \qquad \text{or} \qquad R = k\left(\frac{Et^2}{\rho}\right)^{1/5}.$$

The Taylor article goes to some length to show that $k \approx 1$ and $\rho \approx 1$.

Air has $\rho = 1.2$ kg/m^3 at sea level, and White sands is at 1200 m, which has a density of 1.03 kg/m^3

It follows that

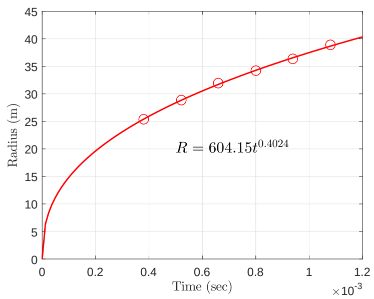
$$R = (Et^2)^{1/5}$$

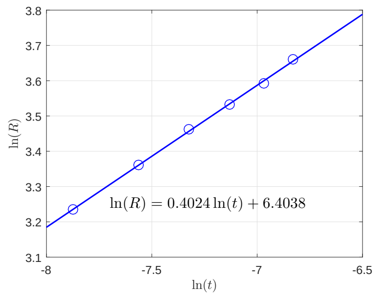
or

$$\ln(R) = \frac{1}{5}\ln(E) + \frac{2}{5}\ln(t).$$

Dimensional Analysis – Atomic Bomb

Below are graphs of the data and the ln of the data:





Dimensional Analysis – Atomic Bomb

From before we have the ***allometric model***

$$R = (Et^2)^{1/5} \qquad \text{or} \qquad \ln(R) = \frac{1}{5}\ln(E) + \frac{2}{5}\ln(t),$$

and the slope of the logarithmic graph from the data agrees with the coefficient obtained by ***dimensional analysis***.

From the data we obtain the intercept, so

$$\frac{1}{5}\ln(E) = 6.4038,$$

which is readily solved for E giving the energy of the atomic blast as

$$E = e^{32.02} = 8.05 \times 10^{13} \text{ J}.$$

Scientists running experiments at the blast site measured the power of the trinity atomic blast as 9×10^{13} J.