Homework – Numerical Quadrature Due Thurs. 11/10/16

Be sure to include all MatLab programs used to obtain answers.

1. a. In the **Lecture Notes** there was an **Open Newton Cotes – Trapezoid Method**, where the interval $[a, b]$ is divided into thirds and function evaluations occur at the middle two points. The integral approximation satisfies:

$$\int_a^b f(x)\,dx = \frac{3h}{2}\Big[f(x_0) + f(x_1)\Big] + \frac{3h^3}{4} f''(\xi),$$

where $h = \frac{b-a}{3}$, $x_0 = a + h$, $x_1 = a + 2h$, and $\xi \in (a, b)$. Create a diagram showing the integral approximation of this method. In the Lecture Notes we verified the coefficients and error terms for the **Trapezoid Rule** and **Simpson's Rule**. With these proofs as a guide, prove that the formula above is correct, *i.e.*, use the Lagrange interpolating polynomials with its error to derive the coefficients and error term specified in the formula above.

b. Use the techniques from the **Lecture Notes** to prove that the **Composite Trapezoid Method** has an error of $\mathcal{O}\left(h^2\right)$. Find the appropriate coefficient on this global error term. Note that n must be a multiple of three, and we subdivide the interval $[a, b]$ into n subintervals with $h = \frac{b-a}{n}$ with the Trapezoid Method applied to each triple of subintervals.

c. Create a MatLab program for the **Composite Trapezoid Method** and numerically verify (as was done in the Lecture Notes) with $f(x) = e^x$ for $x = [0, 6]$ that this Composite Trapezoid Method has convergence $\mathcal{O}\left(h^2\right)$.

d. Modify your code in Part c to approximate

$$\int_0^6 e^x dx$$

to a tolerance of 10^{-6} by starting with $h = 2$ and halving the stepsize until successive approximations are within the tolerance. What stepsize is required for this integral to be accurate to the tolerance of 10^{-6}?

2. Quadrature provides a good tool to approximate π.

a. Use your techniques from Calculus to verify that

$$\pi = \int_{-1}^1 \frac{2}{1+x^2}\,dx.$$

b. Approximate this integral by dividing the interval $[-1, 1]$ into **4** even parts or $h = \frac{1}{2}$. Use the Composite Midpoint, Trapezoid, and Simpson's Rules with this h to give an approximation of π. Give the absolute error with each of the approximations.

c. The Composite Midpoint uses **4** function evaluations, while the other two techniques use **5** function evaluations. Use Gaussian Quadrature with **4** and **5** points to approximate π and find the absolute errors for this integral. Write a short paragraph comparing these four methods.

d. Let $h = \frac{1}{4}$ and repeat the process you did in Part b. Compare this to using Gaussian Quadrature with **9** points. (Hint: You will want to use the `LegendrePolynomials.m` to obtain your quadrature points and weights to perform this approximation.) Again write a short paragraph comparing these four methods.

3. Definite integrals sometimes have the property that the integrand becomes infinite at one or both of the endpoints, but the integral itself is finite. In other words, $\lim_{x\to a} |f(x)| = \infty$ or $\lim_{x\to b} |f(x)| = \infty$, but

$$\int_a^b f(x)dx$$

exists and is finite.

a. Modify our Adaptive Composite Simpson's Rule program (`ASCR.m` and `AdaptiveCSR.m`) so that, if an infinite value of $f(a)$ or $f(b)$ is detected, an appropriate warning message is displayed and $f(x)$ is reevaluated at a point very near to a or b. This allows the adaptive algorithm to proceed and possibly converge.

b. Find an example that triggers the warning, but has a finite integral.

WeBWorK: There are **8** problems in WeBWorK on Lagrange polynomials and Quadrature. With the exception of the first problem all problems have you write something and/or provide MatLab codes.