Homework – Taylor's polynomials Due Thur. 9/7

1. (**Each part is worth 3 pts**) Find the Taylor Series of

   (a) $f(x) = 1/x$ around $x_0 = 1$.

   (b) $f(x) = \sqrt{x}$ around $x_0 = 4$.
   Hint: Identifying patterns is what makes this problem possible. What I mean is that we have

$$f'(x) = \frac{1}{2}x^{-1/2},\ f''(x) = \frac{1}{2}\left(\frac{-1}{2}\right)x^{-3/2},\ f'''(x) = \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)x^{-5/2}$$

   and so we see a pattern emerging that says products of odd numbers in the numerator, and powers of 2 in the denominator. You should convince yourself using induction that

$$\frac{d^j}{dx^j}\sqrt{x} = \frac{(-1)^{j-1}}{2^j}\prod_{k=0}^{j-2}(2k+1)x^{-(2j-1)/2},\ j \geq 2.$$

   Therefore, you would then have

$$\left.\frac{d^j}{dx^j}\sqrt{x}\right|_{x=4} = \frac{(-1)^{j-1}}{2^{3j-1}}\prod_{k=0}^{j-2}(2k+1),\ j \geq 2.$$

2. (**Each part is worth 3 pts**) Find values x_0, δ, and M such that

$$\max_{x\in[x_0-\delta,x_0+\delta]}|f(x)| \leq M$$

for

   (a) $f(x) = \sin(x)/x^3$ for $x \in [1,3]$.
   Hint:

$$|\sin(x)| \leq 1$$

   no matter what x is. So then, we have that

$$\left|\frac{\sin(x)}{x^3}\right| \leq \frac{1}{|x|^3}.$$

   So that means that

$$\max_{x\in[x_0-\delta,x_0+\delta]}\left|\frac{\sin(x)}{x^3}\right| \leq \max_{x\in[x_0-\delta,x_0+\delta]}\frac{1}{|x|^3},$$

   since if we can bound one function in terms of another, then the maximum value of the larger function should be larger than the maximum value of the smaller function. The ultimate strategy here is that dealing with the function $\sin(x)/x^3$ is relatively more difficult than dealing with the function $1/x^3$. Thus, we use a relatively simple bound on the most complicated part of the function, the $\sin(x)$ term, to simplify our lives.

Having done this, you now need to choose a specific number for M, which is an upper bound on $1/|x|^3$, using the fact that $x \in [1,3]$. Once you choose this value for M, you will have that

$$\max_{x\in[x_0-\delta,x_0+\delta]} \left|\frac{\sin(x)}{x^3}\right| \le \max_{x\in[x_0-\delta,x_0+\delta]} \frac{1}{|x|^3} \le M,$$

which simplifies to

$$\max_{x\in[x_0-\delta,x_0+\delta]} \left|\frac{\sin(x)}{x^3}\right| \le M.$$

As for what x_0 and δ are, if we are trying to think in terms of a symmetric interval $[x_0-\delta, x_0+\delta]$, that means x_0 is the midpoint of the interval and δ is the distance from the midpoint to either end of the interval, where our interval is $[1,3]$. Write what your choices are for x_0, δ, and M.

(b) $f(x) = \sqrt{\sin^2(x)+8}$ for $x \in [2,6]$. Here, use the fact that if $y > x$ then $\sqrt{y} > \sqrt{x}$. Then use the same hint and steps as in part (a) of this problem.

Note, this problem is asking you to find reasonable bounds for maxima. This is not about using the derivative to find the maximum, since in many cases, as you can see from the above, this can be difficult to actually do.

3. (**3 pts**) Use the Maclaurin series for e^x, $\cos(x)$, and $\sin(x)$ to demonstrate Euler's formula:

$$e^{ix} = \cos(x) + i\,\sin(x).$$

4. (**Each part is worth 3 pts**) We cannot exactly find the integral:

$$\int_0^1 e^{-x^2}\,dx,$$

but we can approximate it. To do this,

(a) Using a third order Taylor series approximation around $x_0 = 0$, *i.e.,* $T_3(x; x_0 = 0)$, to find an approximation to the integral.

(b) First separate the integral into quarters, *i.e.,* use the fact that

$$\int_0^1 e^{-x^2}\,dx = \int_0^{1/4} e^{-x^2}\,dx + \int_{1/4}^{1/2} e^{-x^2}\,dx + \int_{1/2}^{3/4} e^{-x^2}\,dx + \int_{3/4}^{1} e^{-x^2}\,dx,$$

Then use a third-order Taylor series approximation for each integral. Note, you will need to pick a different value of x_0 for each integral to make doing this worthwhile.

(c) Matlab says that

$$\int_0^1 e^{-x^2}\,dx = 0.746824132812427,$$

Which of your two answers is better? Why?

5. (**Each part is worth 1 pt**) In a Matlab command window, type in the following:

```
X = ['cat'];
Y = ['food'];
```

Write the output from the following commands

(a) `[X Y]`

(b) `[X'; Y']`

(c) `[X' Y']` (and yes, you should get the program yelling at you, why?)

(d) `Y(1:2:4)`

(e) `Y(1:2:end)`

(f) `X(end-1)`

(g) `[Y 'is good']`

(h) `[X [Y 'is good']]`

(i) `length(Y)`

(j) `length(X)`