I, ________________________, pledge that this exam is completely my own work, and that I did not take, borrow or steal any portions from any other person. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

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Signature

Be sure to show all your work or include a copy of your programs.

1. Consider the function given by

$$f(x) = e^{-3x} + 3\sin(x) - 1.$$

a. Write a Taylor polynomial about $x_0 = 0$ of degree 3, $P_3(x)$, and include the remainder term $R_3(x)$ for $x \in [0, 1]$.

b. Use the remainder term to get an upper bound on the error in the approximation, $P_3(x)$, for $x \in [0, 1]$. Find the absolute and relative error between $f(1)$ and $P_3(1)$.

c. One of the roots of this function is $x = 0$. Write Newton's method for finding this root. Show your iterations to a tolerance of 10^{-5}, starting with $x_0 = 0.2$. What is the rate of convergence for this iteration procedure? Explain.

d. Write down a scheme that converges more rapidly than your Newton's method. Show your iterations for this scheme to a tolerance of 10^{-5}, starting with $x_0 = 0.2$. What is its rate of convergence for this iteration procedure? Explain.

e. Use the **secant method** and **Newton's method** to find the other roots of $f(x) = 0$ for $x \in (0, 8]$. Show your iterations and give the rate of convergence near each of the other roots for each method.

2. The integral of the sinc function, $\text{Si}(x)$, also known as the sine integral function, is used in signal processing. It is given by the following formula:

$$\text{Si}(x) = \int_0^x \frac{\sin(u)}{u}\,du.$$

a. Expand $\frac{\sin(u)}{u}$ in a Maclaurin series. Then use this series to integrate term by term to find the Maclaurin series expansion for $\text{Si}(x)$.

b. On the same graph for $x \in [0, 2\pi]$, plot the MatLab sine integral function, `sinint(x)`, with the Taylor polynomials found from Part a of order 3, 5, 7, 9, and 11 ($P_3(x)$, $P_5(x)$, $P_7(x)$,

$P_9(x)$, and $P_{11}(x)$). Be sure to label the axes and provide a legend for the different curves in the graph.

c. Use the expansion in Part a to obtain a bound on the error for $x \in [0, \pi/4]$ for $P_5(x)$, the 5^{th} order polynomial approximation. At $x = \pi$, we find that

$$\mathrm{Si}(\pi) = 1.851937052.$$

Find the relative error at $x = \pi$ for the Taylor polynomials $P_3(\pi)$, $P_5(\pi)$, and $P_7(\pi)$.

3. a. Write a program that generates the iterative sequence

$$y_n = a\, y_{n-1} + b\, y_{n-2}, \quad n \geq 2, \quad a, b \in \mathbb{R}.$$

Your program must take a, b, y_0, y_1, and the maximum value of N as input, and it must produce a plot of the sequence for $0 \leq n \leq N$ with axis labeled semi-log plots, (MatLab `semilogy`). For $N = 20$ and the parameter values:

1. $a = 1$, $b = 1.3$, $y_0 = 1$, $y_1 = 2$,
2. $a = 0.4$, $b = 0.5$, $y_0 = 2$, $y_1 = 5$,

use your program to generate two plots. Explain the behavior you see in the graphs. In particular, find the slope of the lines in your plots in the large N limit both computationally and analytically. (Hint: In order to get the analytic result, you need to use a guess for the solution of the form

$$y_n = \lambda^n.$$

Then use in your guess, and solve for λ.)

b. Take your program in Part a with $a = 1$, $b = 1.3$, $y_0 = 1$, $y_1 = 2$. Modify the code to add every third term ($y_3 + y_6 + y_9 + ...$) as long as the sum is less than M (an input to your program). The program should output the values of $3n$, y_{3n}, and the sum $y_3 + y_6 + y_9 + ... + y_{3n} < M$. Run this program for $M = 2,000,000$, showing the output.

4. The Greek mathematician Archimedes estimated the number π by approximating the circumference of a circle of diameter 1 by the perimeter of both inscribed and circumscribed polygons. The perimeter, s_n, of the inscribed regular polygon with 2^n sides can be given by the recursive formula ($s_n < \pi$):

$$s_{n+1} = \frac{2^{n+1}}{\sqrt{2}} \sqrt{1 - \sqrt{1 - \left(\frac{s_n}{2^n}\right)^2}}, \qquad s_2 = 2\sqrt{2}.$$

a. Write a MatLab program to calculate s_3 to s_{30}. Describe what goes wrong with your calculation and why this occurred.

b. Use some algebra to correct the problem and recompute s_3 to s_{30}.

5. To solve the quadratic equation

$$ax^2 + bx + c = 0,$$

we usually use the "classic" formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1}$$

Explain when this formula suffers from catastrophic cancellation errors on a finite-precision computer.

b. Assume that you have a 4-digit computer (rounding last digit), and the quadratic equation has the parameters $a = 1.000$, $b = -62.74$, and $c = 0.3596$. Find the two real roots for this quadratic equation using your 4-digit computer. Find the actual roots and determine the percent error between the actual roots and the roots found with 4-digit rounding.

c. Derive and explain why and when it is better to use the expression

$$x = \frac{2c}{-b \mp \sqrt{b^2 - 4ac}}. \tag{2}$$

Write a program that computes the roots of the quadratic equation by switching between the "classic" equation (1) and the new formula (2). Clearly explain how you choose the criteria for switching, and develop test cases that show how much more accurate your approach is as opposed to strictly using the classic equation.

d. Repeat Part b with your 4-digit computer using the new formula (2) for the root, which has the least accuracy above. Use the actual value for this root and determine the percent error between this actual root and the root found with 4-digit rounding and formula (2).

6. a. Consider the functions

$$f(x) = 2.6\, e^{0.6x} \qquad \text{and} \qquad g(x) = 3.4\, x^6.$$

These curves intersect when $f(x) = g(x)$ or $F(x) = f(x) - g(x) = 0$. Create programs for the bisection, secant, and Newton's methods to find all roots of $F(x) = 0$ to an accuracy of 10^{-5} for bisection and 10^{-8} for secant and Newton's methods. Describe how you found reasonable initial points for determining your roots. Give your starting values and give the number of iterations until you converge to a root. What criterion do you use for convergence? Briefly discuss the efficiency and stability of each of your programs. Note which program would be your preferred method. Produce a graph or graphs to clearly illustrate all points of intersection for $f(x)$ and $g(x)$, and give both the x and y-values for the points of intersection.

b. Compare the rate of convergence, α, of the three methods using log / log-plots. Explain all details clearly, using the iterations from one of the roots found in Part a.