Homework – Numerical Linear Algebra Due Tues. 10/27/16

Be sure to include all MatLab programs used to obtain answers.

1. (7pts) (From the text, **2.6**.) So, as always, it helps to visualize the matrix A you are dealing with which looks like

$$A = \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & -1 & \cdots & -1 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

Then, in the 1-norm, we have that

$$||A\vec{x}||_1 = \sum_{j=1}^{n-1} \left| x_j - \sum_{k=j+1}^{n} x_k \right| + |x_n|$$

Remember, you want to find

$$||A||_1 = \max_{\vec{x}\neq 0} \frac{||A\vec{x}||_1}{||\vec{x}||_1}.$$

To motivate your thinking, don't forget your friend the triangle inequality

$$|x - y| \leq |x| + |y|.$$

You should readily be able to show then that

$$||A\vec{x}||_1 \leq \sum_{j=1}^{n} \sum_{k=j}^{n} |x_k| \leq n\, ||\vec{x}||_1$$

Then find an $\vec{x}$ to show you can get the inequality to become equality and thus show

$$||A||_1 = n.$$

As for A^{-1}, if you fuss properly you should be able to show that

$$\left|\left|A^{-1}\vec{x}\right|\right|_1 = \sum_{j=1}^{n-1} \left| x_j + \sum_{k=j+1}^{n} 2^{k-(j+1)} x_k \right| + |x_n| \leq 2^{n-1}\, ||\vec{x}||_1 .$$

Again, pick $\vec{x}$ so that the inequality becomes equality, which implies

$$\left|\left|A^{-1}\right|\right|_1 = 2^{n-1}.$$

Then using the definition of condition number implies

$$\kappa(A) = n2^{n-1}.$$

2. (5 pts) (From the text, **2.7**.) Provide two test cases, which show your approach is correct. Note that a good way to produce random $N \times N$ matrices in MatLab is to use the command `randn(N)`. Use this for examples in this problem and the next.

3. (3 pts each) (From the text, **2.18 a,b**.) Work only Parts a and b, skipping Part c. Again you'll want to use a collection of random matrices.

4. (12 pts) In class we solved $Ax = b$ using a **Direct Method**, specifically Gaussian elimination with partial pivoting. This used a fair number of steps, which went up computationally like n^3, where $n \times n$ was the dimension of the matrix A. Often large matrices, especially **sparse** matrices are solved iteratively. These iterative methods are particularly valuable when the matrix is **diagonal dominant**, which is where the value of the diagonal elements of A exceed the sum of the other elements in the row.

a. Suppose the matrix A is written:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

and split this matrix into a diagonal matrix D, a lower triangular matrix L, and an upper triangular matrix U. Write the matrix:

$$\begin{aligned} A &= \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{pmatrix} - \begin{pmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -a_{n-1,n} \\ 0 & \cdots & 0 & 0 \end{pmatrix} \\ &= D - L - U \end{aligned}$$

Show that if D^{-1} exists ($a_{ii} \neq 0$ for $1 \leq i \leq n$), then the equation

$$Ax = (D - L - U)x = b$$

has the solution

$$x = D^{-1}(L + U)x + D^{-1}b.$$

b. If $T = D^{-1}(L + U)$ and $c = D^{-1}b$, then**Jacobi's iterative method** satisfies the formula

$$x^{(k)} = D^{-1}(L + U)x^{(k-1)} + D^{-1}b = Tx^{(k-1)} + c.$$

Consider the system:

$$\begin{pmatrix} 4 & 1 & -1 & 1 \\ 1 & 4 & -1 & -1 \\ -1 & -1 & 5 & 1 \\ 1 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Find T and c. Let $x^{(0)} = \mathbf{0}$, the zero vector, and find the first **3** iterations using Jacobi's iterative method.

c. Continue Jacobi's iterative method for **10** iterations. Give the value of $x^{(10)}$ Also, find the exact solution, x^*, using Matlab's `linsolve` program. Determine the error between the 10^{th} iterate and the exact solution in the 1-norm, 2-norm, and ∞-norm, *i.e.*, find

$$\|x^{(10)} - x^*\|_1 \quad \text{and} \quad \|x^{(10)} - x^*\|_2 \quad \text{and} \quad \|x^{(10)} - x^*\|_\infty.$$

WeBWorK: There are **4** problems in WeBWorK where you perform LU-factorization. However, you do NOT use partial pivoting as we did in class for the most stable routine. There are no written parts in these problems. They are computational to allow you to demonstrate that you know the basics to LU-factorization.