Math 541 - Numerical Analysis

Lecture Notes – MatLab Programming

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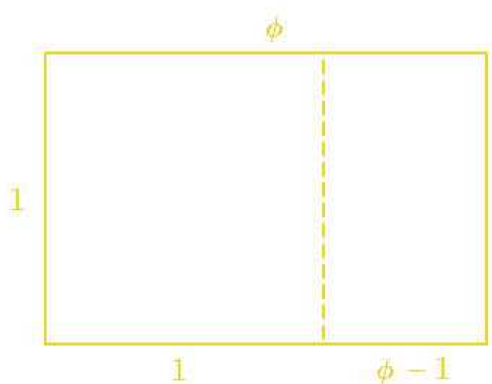
Golden Rectangle

Since ancient times, the Golden Rectangle has been believed to be aesthetically pleasing, where the large rectangle ($\phi \times 1$) composed of a unit square and smaller rectangle is *similar* to the adjacent smaller rectangle ($1 \times (\phi - 1)$)

The Golden Rectangle is shown below



Golden Ratio

The Golden Ratio satisfies

$$\frac{1}{\phi} = \frac{\phi - 1}{1}$$

It follows that ϕ satisfies

$$\phi^2 - \phi - 1 = 0,$$

which by the quadratic equation gives

$$\phi = \frac{1 \pm \sqrt{5}}{2}$$

Since only the positive root makes sense

$$\phi = \frac{1 + \sqrt{5}}{2}$$

Golden Ratio and MatLab

MatLab: We explore MatLab commands

- Numbers in MatLab:
  `phi = (1 + sqrt(5))/2`
  - Default is 4 places after decimal
  - Obtain 16 digits (**MatLab** standard) with
    `format long` or `double(phi)`
  - Obtain 50 digits with
    `vpa(phi,50)`
- The polynomial is $x^2 - x - 1$, which is solved by
  `p = [1 -1 -1]`
  `r = roots(p)`
- Symbolic package to solve quadratic:
  `syms x`
  `r = solve(1/x == x-1)`

Basic MatLab – Function and Graph

MatLab: Define a basic function and study

- Create an ***anonymous function***
  `f = @(x)1./x-(x-1)`
- Create a simple plot of the function: `ezplot(f, 0, 4)`
  Note the plot selects a default reasonable range
- Find the zero of a function
  `phi = fzero(f, 1)`
- Add this point to the graph
  `hold on`
  `plot(phi,0,'o')`
- Use the **fsolve** routine from the Optimization package
  `fsolve(@(x)f(x), 1)`

MatLab Script for Graph

Create **MatLab Scripts** to make graphs (modify old ones)

```
% GOLDRECT Plot the golden rectangle

phi = (1 + sqrt(5))/2;
x = [0 phi phi 0 0];
y = [0 0 1 1 0];
u = [1 1];
v = [0 1];
plot(x,y,'b',u,v,'b--')
text(phi/2, 1.05,'$\phi$','interpreter','latex')
text((1+phi)/2, -0.05, '$\phi - ...
    1$','interpreter','latex')
text(-0.05,0.5,'1')
text(0.5,-0.05,'1')
axis equal
axis off
set(gcf,'color','white')
```


MatLab Function

The Golden fraction is a continued fraction of the form:

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}$$

- Create a **MatLab Function** that produce this continued fraction to n terms
- Display the continued fraction to n terms
- Compute the fractional value, then show the decimal value of this
- Find the error of truncating after n terms

MatLab Function

```
function goldfract(n)
%GOLDFRACT  Golden ratio continued fraction
%   GOLDFRACT(n) displays n terms
p = '1';
for k = 1:n
    p = ['1+1/(' p ')'];
end
p
p = 1; q = 1;
for k = 1:n
    s = p;
    p = p + q;
    q = s;
end
p = sprintf('%d/%d',p,q)
format long; p = eval(p)
format short; err = (1 + sqrt(5))/2 - p
end
```


Fibonacci Numbers

Fibonacci Numbers appear often in **Nature** (hyperlink), petals of flowers, organization of pine cones and sunflowers, branching trees and bones, shell spirals, reproduction, etc.

Leonardo Fibonacci originally posed the following:

> *A man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?*

Malthusian growth, where immediate fertility is assumed, gives the **difference equation**

$$P_{n+1} = (1 + r)P_n,$$

which is easily solved, given an initial population, P_0

Fibonacci Numbers

Fibonacci's problem introduces a delay for maturation, complicating the problem

If f_n denotes the number of pairs of rabbits after n months, then the number of pairs is the number at the beginning plus the number of births or the **difference equation** becomes:

$$f_n = f_{n-1} + f_{n-2}$$

with initial conditions, $f_0 = 1$ and $f_1 = 1$, so

$$f_2 = f_1 + f_0 = 1 + 1 = 2$$

and

$$f_3 = f_2 + f_1 = 2 + 1 = 3, \qquad f_4 = f_3 + f_2 = 3 + 2 = 5$$

Fibonacci Program

We assume that we start with **one pair** of rabbits ($f_0 = 1$)
They don't reproduce the first month, so there is still one pair ($f_1 = 1$)

We write a MatLab program for the **Fibonacci sequence**

```
function f = fibonacci(n)
%FIBONACCI: Fibonacci sequence
%   f = FIBONACCI(n)generates the first n ...
    Fibonacci numbers
f = zeros(n,1);  % (n x 1) matrix of zeros
f(1) = 1;
f(2) = 2;
for k = 3:n
    f(k) = f(k-1) + f(k-2);
end
end
```

Fibonacci vs Malthusian Growth

Fibonacci asked how many pairs of rabbits after 12 months, so invoking the program:
`fibonacci(12)`
shows the sequence
`1 2 3 5 8 13 21 34 55 89 144 233`

Malthusian growth doubles every month, so MatLab computes with
`2.^(1:12)`
resulting in
`2 4 8 16 32 64 128 256 512 1024 2048 4096`

It follows that the delay in maturation results in 233 pairs of rabbits at the end of the year, compared to 4096 without the maturation

Fibonacci Program

The program below uses a `while` loop to determine how long it takes to pass a given number in a Fibonacci sequence

```
function f = fibomax(M)
%FIBONACCI: Fibonacci sequence
%   f = FIBOMAX(M)generates the Fibonacci numbers
%   until just passing a specified integer M
f(1) = 1;
f(2) = 2;
k = 2;
while (f(k) ≤ M)
    f(k+1) = f(k) + f(k-1);
    k = k + 1;
end
j = k - 1; F = f(j);
sprintf('Last Fibonacci number before %d is %d ...
    and occurs after %d months',M,F,j)
end
```

the population of 4096, while is the **Malthusian growth** population

Fibonacci Program

The previous **MatLab** program, `fibomax(4096)` shows it takes 18 months to surpass the population of 4096

Specifically, it shows that the 17^{th} Fibonacci number is 2584, while the 18^{th} Fibonacci number is 4181

The **Malthusian growth** population model (doubling monthly) gave 4096 after 12 months, thus the Fibonacci delayed growth model increases more slowly

The use of **MatLab** `while` loops is valuable for stopping when a condition is met

Fibonacci and Golden Ratio

The Golden ratio $\phi = \frac{1+\sqrt{5}}{2} = 1.618033988749895$

Interestingly, the ratio of successive **Fibonacci numbers** in the limit tends to ϕ

The program above in MatLab computes the ratio of the first 39 ratios:
`n = 40; f = fibonacci(n):`
`f(2:n)./f(1:n-1)`

The ratios at $n = 5$, 10, 20, and 38 are

$$\frac{f(5)}{f(4)} = 1.6 \qquad \frac{f(10)}{f(9)} = 1.618181818181818$$

$$\frac{f(20)}{f(19)} = 1.618033998521803 \qquad \frac{f(38)}{f(37)} = 1.618033988749895$$

The last ratio agrees with the Golden ratio to double precision

Collatz Problem

The **Collatz Problem** (hyperlink) is an unsolved problem in **Number Theory**

The problem is an easily stated sequence of events

- If $n = 1$, stop.
- If n is even, replace it with $n/2$.
- If n is odd, replace it with $3n + 1$.

Depending on the choice of n, it is conjectured that this sequence always terminates after a finite number of steps

We won't try to prove this conjecture, but write a program to follow the sequence for any n

It happens that roundoff error creates problems if an element in the sequence exceeds 2^{53}

$3n + 1$ Sequence

This problem has an iterative process with conditional statements for deciding what to do

This is readily programmed using **MatLab** `if` and `while` statements (See program below)

```
function y = f_3nplus1(n)
% ``Three n plus 1''.
% Study the 3n+1 sequence.
y = n;
while n > 1
   if rem(n,2)==0   % Could use mod(n,2)==0
      n = n/2;      % Even numbers/2
   else
      n = 3*n+1;    % Odd numbers*3 + 1
   end
   y = [y n];
end
end
```


$3n + 1$ Sequence

Implementation of the previous program produces a sequence ending in 1

`f_3nplus1(7)` yields:
`7 22 11 34 17 52 26 13 40 20 10 5 16 8 4 2 1`

`f_3nplus1(17)` yields:
`17 52 26 13 40 20 10 5 16 8 4 2 1`

This program quickly computes the sequence

Other programs could be written to study large sets of numbers to try to see patterns

The Lecture Note page provides a GUI program from the text for more elaborate **MatLab** code of this problem

The Number e

The Maclaurin series gives $e = \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!}$

```
function eapprox = ecomp(N)
%Approximation to e with N terms
eapprox = 1;
term = 1; tic
for jj = 1:N
    term = term./jj;
    eapprox = eapprox + term;
end
toc
end
```

Executing the program
`ecomp(10)`
yields $e \approx 2.718281801146385$, while $e = 2.718281828459046$

The Number e

MatLab has a compact way of writing sums (adding components of a vector)

`eapprox = sum(1./factorial(0:N))`

- Computers cannot perform infinite sums, so we must terminate a program
- The program above uses a `for` loop, and we select the number of terms
- Numerically, we usually want a certain accuracy
- This implies setting a tolerance and running a loop until the error in computation is below a set tolerance
- Below is a program for computing e by truncating the series at a particular tolerance at the last term

The Number e

```
function eapprox = esum(tol)
%Approximation to e to a given tolerance
eapprox = 1;
term = eapprox;
n = 1;
while (abs(term) ≥ tol)
    term = term/n;              % New term from old/n
    eapprox = eapprox + term; % Sum adds new term
    n = n + 1;
end
end
```

Executing the program

`esum(1e-4)`

yields $e \approx 2.718278769841270$, while $e = 2.718281828459046$

Exponential Function, e^x

The **Maclaurin series** of e^x satisfies $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

- Can we write a program for an x that gives e^x to a certain tolerance?
- The **ratio test** gives convergence of this series for all x

$$\frac{|x|^{k+1}}{(k+1)!} \cdot \frac{k!}{|x|^k} = \frac{|x|}{k+1} < 1$$

- For any fixed x, then k can be taken large enough that the **ratio test** is satisfied
- After a large enough k, then for the given x the terms of the series monotonically decrease (in absolute value) toward zero
- Like the previous program, once one term is below the tolerance, then all subsequent terms would be below the tolerance

Exponential Function, e^x

```
function esumx = exp_sum(x,tol)
%Approximation to e^x to a given tolerance
esumx = 1;
term = 1;
k = 1;
while (max(abs(term)) ≥ tol)
    term = term.*x./k;     % Recursive formula
    esumx = esumx + term; % Sum adds new term
    k = k + 1;
end
end
```

Executing the program

`exp_sum([-1 0 1 2 4],1e-4)`

yields

`0.3678794 1.0000000 2.7182818 7.3890561 54.5981500`

Exponential Function, e^x

Important Points in the program:

- **Recursive formula**, where ***new term*** builds off of ***old term***
- Series expansion:

$$e^x = 1 + 1 \cdot x + x \cdot \frac{x}{2} + \frac{x^2}{2!} \cdot \frac{x}{3} + \frac{x^3}{3!} \cdot \frac{x}{4} + \dots$$

- Each iteration through loop adds the ***new term*** to the ***sum***
- The ***new terms*** are compared the user supplied ***tolerance***
- Provided the ***new term*** is greater than the ***tolerance*** the sum is increased, otherwise the program terminates
- This program is set up for **vector** x, so the new terms and the sums are vectors
- Want all ***new terms*** less than the ***tolerance*** before exiting program

Graph of Exponential Function, e^x

```
function exp_plot(Lx,res,tol)
% Create a plot using the exp_sum function
tic
xx = linspace(-Lx,Lx,res);  % x points of evaluation
yy = exp_sum(xx,tol);        % run exp series program
yy1 = exp(xx);              % evaluate e^x with MatLab

figure(101)                  % assign a figure number

plot(xx,yy,'r-','LineWidth',1.5);   % plot the series
hold on                              % add more graphs
plot(xx,yy1,'b--','LineWidth',1.5);% plot e^x
grid;                                % provide gridlines
```

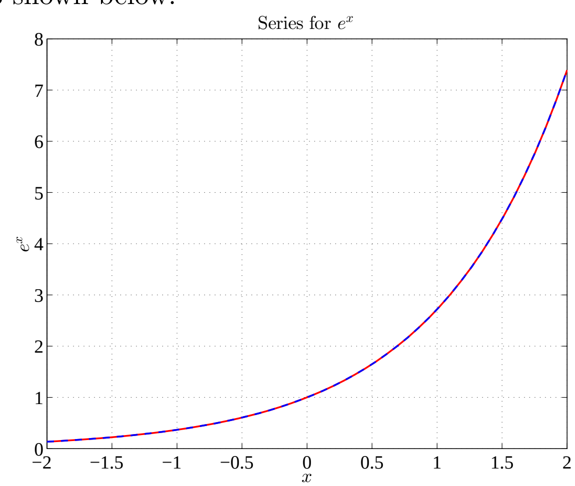

Graph of Exponential Function, e^x

```
% Set up fonts and labels for the Graph
fontlabs = 'Times New Roman';
xlabel('$x$','FontSize',16,'FontName',fontlabs, ...
    'interpreter','latex');
ylabel('$e^x$','FontSize',16,'FontName',fontlabs, ...
    'interpreter','latex');
mytitle = 'Series for $e^x$';
title(mytitle,'FontSize',16,'FontName', ...
    'Times New Roman','interpreter','latex');
set(gca,'FontSize',16);

%print -depsc exp_gr.eps       % Create EPS file ...
    (Figure)
end
```


Graph of Exponential Function, e^x

Result of the previous program with $Lx = 2$, $res = 100$, and $tol = 1e-3$ shown below:



Bessel Functions

Bessel's Equation is an important differential equation:

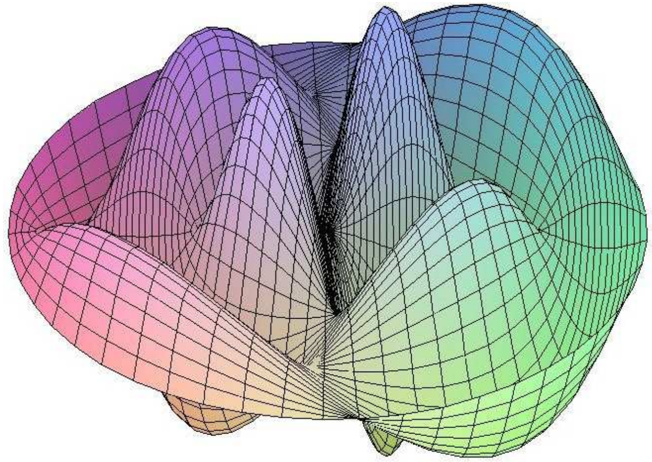
$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - m^2)y = 0$$

- This equation appears naturally in applied problems with circular or cylindrical geometry - vibrating drums, aircraft design, ...
- Equation is a regular singular differential equation
- Solved using **Method of Frobenius** – A power series technique
- Series solution, m^{th} order Bessel function of 1^{st} kind (m an integer) is given by:

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(m+n)!}\left(\frac{x}{2}\right)^{2n+m}$$

Bessel Function, $J_m(x)$

Figure showing a vibrating membrane, which uses a **Bessel function**



m^{th} Order Bessel Function

The m^{th} **Order Bessel Function of 1^{st} Kind** satisfies:

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(m+n)!}\left(\frac{x}{2}\right)^{2n+m}$$

- Must find a **recursive relationship**
- Create appropriate stopping criteria with `while` loop based on a **tolerance**
- Include a condition to avoid too many steps
- **Ratio Test** shows this series converges for all x

m^{th} Order Bessel Function

Find **recursive relation**:

$$\begin{aligned}
J_m(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(m+n)!}\left(\frac{x}{2}\right)^{2n+m} \\
&= \frac{1}{m!}\left(\frac{x}{2}\right)^m - \frac{1}{1!(m+1)!}\left(\frac{x}{2}\right)^{(m+2)} + \frac{1}{2!(m+2)!}\left(\frac{x}{2}\right)^{(m+4)} - \dots \\
&= \frac{1}{m!}\left(\frac{x}{2}\right)^m + \frac{1}{m!}\left(\frac{x}{2}\right)^m \cdot \frac{(-1)}{(m+1)}\left(\frac{x}{2}\right)^2 \\
&\quad + \frac{1}{1!(m+1)!}\left(\frac{x}{2}\right)^{(m+2)} \cdot \frac{1}{2(m+2)}\left(\frac{x}{2}\right)^2 + \dots
\end{aligned}$$

Thus,

```
New Term = Old Term.*(-(x/2).^2/((n+1)*(n+m+1)))
```

Bessel Function, $J_m(x)$

Below is the **MatLab** code for $J_m(x)$

```
function tot = bessel(x,m,tol)
% Approximation to J_m(x) to a given tolerance
tot = (x/2).^m/factorial(m);
term = tot;
k = 1;
while (max(abs(term)) ≥ tol)
    term = term.*(-(x/2).^2/(k*(k+m)));
    tot = tot + term;
    k = k + 1;
end
end
```


Bessel Function, $J_m(x)$

Below is the **MatLab** code for $J_m(x)$ with a maximum number of steps

```
function tot = bessel2(x,m,tol,Nmax)
% Approximation to J_m(x) to a given tolerance
% or Nmax terms
tot = (x/2).^m/factorial(m);
term = tot;
k = 1;
while ((max(abs(term)) ≥ tol)&& (k ≤ Nmax))
    term = term.*(-(x/2).^2/(k*(k+m)));
    tot = tot + term;
    k = k + 1;
end
end
```


Graph of $J_m(x)$

Next 2 slides show Bessel plot routine, then a graph

```
function bessel_plot(m,Lx,res,tol)
% Create a plot using the bessel function

xx = linspace(0,Lx,res); % x points of evaluation
yy = bessel(xx,m,tol);   % run bessel series program

figure(101)                  % assign a figure number

plot(xx,yy,'r-','LineWidth',2.0);  % plot the series
grid;                              % provide gridlines
```

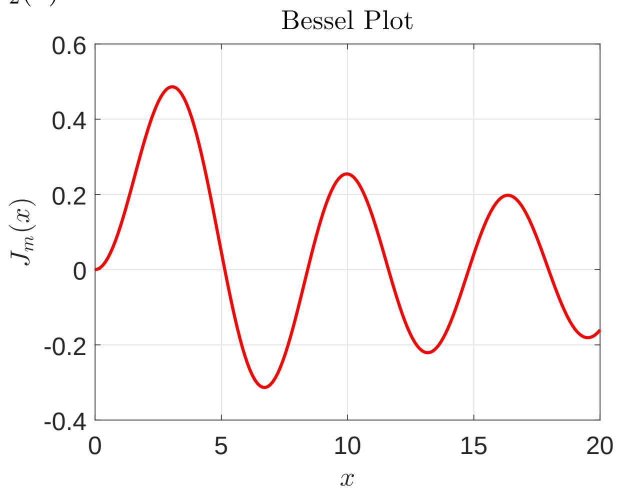

Graph of $J_m(x)$

```
% Set up fonts and labels for the Graph
fontlabs = 'Times New Roman';
xlabel('$x$','FontSize',16,'FontName',fontlabs, ...
    'interpreter','latex');
ylabel('$J_m(x)$','FontSize',16,'FontName',fontlabs, ...
    ...
    'interpreter','latex');
mytitle = 'Bessel Plot';
title(mytitle,'FontSize',16,'FontName', ...
    'Times New Roman','interpreter','latex');
set(gca,'FontSize',16);

print -depsc bessel_gr.eps     % Create EPS file ...
    (Figure)
end
```

Graph of $J_2(x)$

Graph of $J_2(x)$



Graph of Multiple Bessel Functions

```
function bessel2_plot(Lx,res,tol)
% Create a plot using the bessel function

xx = linspace(0,Lx,res); % x points of evaluation
yy = bessel(xx,0,tol);   % run bessel series program

figure(102)                % assign a figure number

plot(xx,yy,'k-','LineWidth',2.0);  % plot the series
hold on
grid;       % provide gridlines
```


Graph of Multiple Bessel Functions

```
for m = 1:3
    if (m == 1)
        lsty = 'b-';
    elseif (m == 2)
        lsty = 'r-';
    else
        lsty = 'm-';
    end
    yy = bessel(xx,m,tol);
    plot(xx,yy,lsty,'LineWidth',2.0);
end
h = legend('$J_0(x)$', '$J_1(x)$', '$J_2(x)$', ...
    '$J_3(x)$',...
    'Location','northeast');
 set(h,'Interpreter','latex')
```


Graph of Multiple Bessel Functions

```
% Set up fonts and labels for the Graph
fontlabs = 'Times New Roman';
xlabel('$x$','FontSize',16,'FontName',fontlabs, ...
    'interpreter','latex');
ylabel('$J_m(x)$','FontSize',16,'FontName',fontlabs, ...
    ...
    'interpreter','latex');
mytitle = 'Bessel Plot';
title(mytitle,'FontSize',16,'FontName', ...
    'Times New Roman','interpreter','latex');
set(gca,'FontSize',16);

print -depsc bessel2_gr.eps    % Create EPS file ...
    (Figure)
end
```

Graph of Multiple Bessel Functions

Graph of $J_0(x)$, $J_1(x)$, $J_2(x)$, and $J_3(x)$