Math 541 - Numerical Analysis

Planets Example

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Planets Example

This problem examines ***Kepler's Third Law***.

We use the ***power law*** to determine the ***period of orbit*** about the **sun** for all ***planets*** given information about some of the planets.

Let d be the mean distance $\times 10^6$ km from the sun and p be the period of orbit in (Earth) days about the sun.

Planet	Distance d	Period p
Mercury	57.9	87.96
Earth	149.6	365.25
Jupiter	778.3	4337
Neptune	4497	60200

Allometric Model

We seek an ***Allometric model*** of the form:

$$p = kd^a.$$

By taking ***logarithms***, we have:

$$\ln(p) = \ln(k) + a\ln(d),$$

which is ***linear*** in the logarithms of the data.

Below is the Table of the ***logarithms of the data***.

Planet	$\ln(d)$	$\ln(p)$
Mercury	4.058717	4.476882
Earth	5.007965	5.900582
Jupiter	6.657112	8.374938
Neptune	8.411166	11.005428

Direct Linear Least Squares

The logarithmic model is:

$$\ln(p) = \ln(k) + a\ln(d),$$

so if $P_i = \ln(p_i)$, $D_i = \ln(d_i)$, and $K = \ln(k)$, the **Linear Least Squares model**, $P = K + a\,D$, satisfies the error:

$$E(K,a) = \sum_{i=0}^{n} [(K + a\,D_i) - P_i]^2$$

This is minimized for data set $(D_i, P_i), i = 0, ..., 3$.

Earlier we saw an easy formulation without matrices:

Define the averages

$$\bar{D} = \frac{1}{4}\sum_{i=0}^{3} D_i \qquad \text{and} \qquad \bar{P} = \frac{1}{4}\sum_{i=0}^{3} P_i.$$

The best fitting slope and intercept are

$$a = \frac{\sum_{i=0}^{3}(D_i - \bar{D})P_i}{\sum_{i=0}^{3}(D_i - \bar{D})^2} \qquad \text{and} \qquad K = \bar{P} - a\bar{D}.$$

Direct Linear Least Squares

The ***Allometric model*** is:

$$p = k\, d^a.$$

The averages ($\ln(d_i)$ and $\ln(p_i)$) are

$$\bar{D} = \frac{1}{4}\sum_{i=0}^{3} D_i = 6.03374 \qquad \text{and} \qquad \bar{P} = \frac{1}{4}\sum_{i=0}^{3} P_i = 7.43946.$$

The best fitting slope is

$$\begin{aligned} a &= \frac{\sum_{i=0}^{3}(D_i - \bar{D})P_i}{\sum_{i=0}^{3}(D_i - \bar{D})^2} \\ &= \frac{(4.05872 - \bar{D})4.47688 + (5.00797 - \bar{D})5.90058 + (6.65711 - \bar{D})8.37494 + (8.41117 - \bar{D})11.00543}{(4.05872 - \bar{D})^2 + (5.00797 - \bar{D})^2 + (6.65711 - \bar{D})^2 + (8.41117 - \bar{D})^2} \\ &= 1.50001. \end{aligned}$$

The best intercept is

$$\ln(k) = K = \bar{P} - a\bar{D} = 7.43946 - 1.50001(6.03374) = -1.61124,$$

so $k = 0.199639$, giving ***Kepler's Law***:

$$p = 0.199639 d^{1.5}.$$