Math 541 - Numerical Analysis

Lecture Notes – Linear Algebra: Part B

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Outline

Errors

Errors: Consider the system

$$Ax = b$$

- The coefficients in A and values in b are rarely known exactly
- **Experimental** (***observational***) and **round-off errors** enter almost every system
- How much effect is there from perturbations to the system?
- Problems arise when A is ***singular*** or ***nearly singular***
- ***Singular matrices*** result in either **no solution** to the system or the **solution is not unique**
- If A is near the identity, then small changes in b result in small changes in x

Roundoff Errors

Consider the system

$$Ax = b \qquad \text{with} \quad x = A^{-1}b$$

- Almost always some ***computed error*** – Denote this by x_*
  - The **error** is given by
    $$e = x - x_*$$
  - The **residual** is given by
    $$r = b - Ax_*$$
- If either error is zero, theory gives the other being zero
- What if one of the errors is small?

Roundoff Error – Example

Example: Perform **3-digit** arithmetic on the system:

$$\begin{pmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.217 \\ 0.254 \end{pmatrix}$$

Perform **Gaussian Elimination** with **partial pivoting**, beginning with $R_1 \longleftrightarrow R_2$

The multiplier is $m_1 = \frac{0.780}{0.913} = 0.854$ and the operation is $R_2 \rightarrow R_2 - m_1 R_1$, resulting in

$$\begin{pmatrix} 0.913 & 0.659 \\ 0 & 0.001 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.254 \\ 0.001 \end{pmatrix}$$

This gives

$$x_2 = \frac{0.001}{0.001} \qquad \text{and} \qquad x_1 = \frac{0.254 - 0.659}{0.913} = -0.443$$

Roundoff Error – Example

Example from before is:

$$\begin{pmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.217 \\ 0.254 \end{pmatrix}$$

and **3-digit arithmetic** gave the solution

$$x_1 = -0.443 \qquad \text{and} \qquad x_2 = 1.000$$

The ***residual*** is easily calculated

$$r = b - Ax_* = \begin{pmatrix} 0.217 - 0.780(-0.443) + 0.563(1.000) \\ 0.254 - 0.913(-0.443) + 0.659(1.000) \end{pmatrix} = \begin{pmatrix} -0.000460 \\ -0.000541 \end{pmatrix},$$

which has each component $< 10^{-3}$

It is easy to see that the actual solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1.000 \\ -1.000 \end{pmatrix},$$

which is far from the computed solution

Roundoff Error – Example

In this **Example**, the matrix is close to ***singular***, so not typical. With 6 or more digits, the **Gaussian Elimination** with **partial pivoting** gives

$$\begin{pmatrix} 0.913000 & 0.659000 \\ 0 & -0.000001 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.254000 \\ 0.000001 \end{pmatrix}$$

and the correct solution arises

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1.00000 \\ -1.00000 \end{pmatrix}$$

Because of the ***singularity*** almost any answer could arise for x_2.

The computations produce a very small ***residual***, and the two equations are very close to each other.

If ***singular***, then a single equation satisfies the problem.

Thus, the first equation of the original problem is nearly parallel and close to the first equation of the problem with a little error.

Norms

Definition (l_p Norm)

Consider an n-dimensional vector $x = [x_1, ..., x_n]^T$. The l_p **norm** for the vector x is defined by the following:

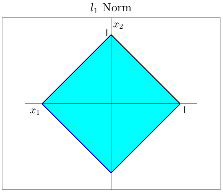
$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

Almost always the norms use $p = 1$, $p = 2$ (Euclidean or distance), or $p = \infty$ (max)

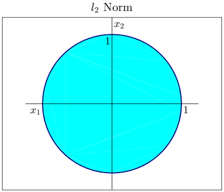
For $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we have $\|x\|_2 = \left(x_1^2 + x_2^2\right)^{1/2}$

Unit Circles

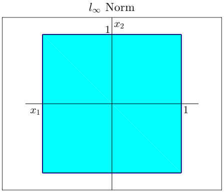
Consider $\|x\| \le 1$ in different **norms**



$$\|x\|_1 \le 1$$
or
$$|x_1| + |x_2| \le 1$$



$$\|x\|_2 \le 1$$
or
$$\left(|x_1|^2 + |x_2|^2\right)^{1/2} \le 1$$



$$\|x\|_\infty \le 1$$
or
$$\max\{|x_1|, |x_2|\} \le 1$$

Norms

Let $x = [x_1, ..., x_n]^T \in \mathbb{R}^n$, then the **norms** for $p = 1$, $p = 2$, or $p = \infty$ satisfy:

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^{n} |x_i| \\ \|x\|_2 &= \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \\ \|x\|_\infty &= \max_i \{|x_i|\} \end{aligned}$$

Property (Norm)

Given an n-dimensional vector $x = [x_1, ..., x_n]^T$, then:

$$\begin{aligned} \|x\| > 0, &\qquad \text{if } x_i \neq 0 \text{ for some } i, \\ \|x\| = 0, &\qquad \text{if } x_i = 0 \text{ for all } i. \end{aligned}$$

Norm – Example

Example: Consider $x = [0.2, 0.4, 0.6, 0.8]$.

- For $p = 1$,

$$\|x\|_1 = \sum_{i=1}^{4} |x_i| = 0.2 + 0.4 + 0.6 + 0.8 = 2.0$$

- **MatLab** command is `norm(x,1)`
- For $p = 2$,

$$\|x\|_2 = \left(\sum_{i=1}^{4} |x_i|^2 \right)^{1/2} = \sqrt{0.04 + 0.16 + 0.36 + 0.64} = 1.0954$$

- **MatLab** command is `norm(x)` or `norm(x,2)`
- For $p = \infty$,

$$\|x\|_\infty = \max_i |x_i| = 0.8$$

- **MatLab** command is `norm(x,inf)`

Norm of a Matrix

Definition (Matrix Norm)

A **matrix norm** on the set of all $n \times n$ matrices is a real-valued function, $\| \cdot \|$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:

1. $\|A\| \geq 0$;
2. $\|A\| = 0$, if and only if A is $\mathbf{0}$, the matrix with all entries 0;
3. $\|\alpha A\| = |\alpha|\|A\|$;
4. $\|A + B\| \leq \|A\| + \|B\|$;
5. $\|AB\| \leq \|A\|\|B\|$;

Norm of a Matrix

Norm of a Matrix: There are a number of norms on a matrix. The most common norm for a matrix is defined by the vector norms for $\mathbb{R}^n$

Theorem (Matrix Norm)

If $\|\cdot\|$ is a vector norm on $\mathbb{R}^n$, then

$$\|A\| = \max_{\|x\|=1} \|Ax\| = \max_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|}$$

is a matrix norm.

It follows that for any x

$$\|A\| \geq \frac{\|Ax\|}{\|x\|} \qquad \text{or} \qquad \|Ax\| \leq \|A\|\|x\|$$

Range of Changes for a Matrix

As x is varied, Ax varies. The ***range*** satisfies:

$$\begin{aligned} M &= \max_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|} \\ m &= \min_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|} \end{aligned}$$

Note: If A is ***singular***, then $m = 0$

Definition (Condition Number)

Let M and m be defined as above for a matrix A. The **Condition Number** for A is given by:

$$\kappa(A) = \frac{M}{m}$$

By the definition it follows that $\kappa \geq 1$

Rounding Error

Consider a system of equations

$$Ax = b$$

Assume that A is known, and there is an **error** in x with $x + \delta x$

$$A(x + \delta x) = b + \delta b$$

This assumes a ***roundoff error*** in x and the result of A acting on x produces a ***roundoff error*** δb

With norms we see for $A(\delta x) = \delta b$

$$\|A(\delta x)\| = \|\delta b\| \qquad \text{or} \qquad \|\delta b\| \le \|A\|\|\delta x\|$$

or

$$\|\delta x\| \ge \frac{1}{\|A\|}\|\delta b\|$$

Condition Number

If A is **invertible**, then $\delta x = A^{-1}(\delta b)$

$$\|\delta x\| = \|A^{-1}(\delta b)\| \leq \|A^{-1}\| \|\delta b\|$$

Definition (Alternate: Condition Number)

The **Condition Number** for a matrix A satisfies:

$$\kappa(A) = \|A\| \|A^{-1}\|.$$

The **Condition Number**, $\kappa(A)$, measures how much impact ***roundoff*** has

Error in Solution

For the perturbation,

$$\|\delta x\| = \|A^{-1}(\delta b)\| \le \|A^{-1}\|\|\delta b\| = \kappa(A)\frac{\|\delta b\|}{\|A\|}$$

From $Ax = b$, we have $\|b\| \le \|A\|\|x\|$ or $\frac{1}{\|A\|} \le \frac{\|x\|}{\|b\|}$

It follows that

$$\|\delta x\| \le \kappa(A)\frac{\|\delta b\|}{\|A\|} \le \kappa(A)\|\delta b\|\frac{\|x\|}{\|b\|}$$

Rearranging

$$\frac{\|\delta x\|}{\|x\|} \le \kappa(A)\frac{\|\delta b\|}{\|b\|}$$

The ***relative error*** in b gives the ***relative error*** provided $\kappa(A) \approx 1$

Roughly, $\kappa(A) = \|A^{-1}\|\|A\|$ measures the ***relative error*** **magnification factor**

Properties

Some properties of the **Condition Number**

Property (Basic Properties)

1. *The **Condition Number** satisfies*

$$\kappa(A) \geq 1$$

2. *If P is a **Permutation Matrix**, then*

$$\kappa(P) = 1$$

3. *If D is a **Diagonal Matrix**, then*

$$\kappa(D) = \frac{\max |d_{ii}|}{\min |d_{ii}|}$$

Example with l_1-norm

Example with l_1-norm: Let

$$A = \begin{pmatrix} 4.1 & 2.8 \\ 9.7 & 6.6 \end{pmatrix}, \qquad b = \begin{pmatrix} 4.1 \\ 9.7 \end{pmatrix}, \qquad x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Clearly, $Ax = b$ with

$$\|b\| = 13.8 \qquad \text{and} \qquad \|x\| = 1$$

Make a small perturbation, so

$$\bar{b} = \begin{pmatrix} 4.11 \\ 9.70 \end{pmatrix}$$

The solution becomes

$$\bar{x} = \begin{pmatrix} 0.34 \\ 0.97 \end{pmatrix}$$

Let $\delta b = b - \bar{b}$ and $\delta x = x - \bar{x}$, then

$$\|\delta b\| = 0.01 \qquad \text{and} \qquad \|\delta x\| = 1.63$$

Example with l_1-norm

Example with l_1-norm: From before

$$\|\delta b\| = 0.01 \qquad \text{and} \qquad \|\delta x\| = 1.63$$

Thus, a small perturbation in b completely changes x

The relative changes are

$$\frac{\|\delta b\|}{\|b\|} = 0.0007246 \qquad \text{and} \qquad \frac{\|\delta x\|}{\|x\|} = 1.63$$

Because $\kappa(A)$ is the ***maximum magnification factor***

$$\kappa(A) \geq \frac{1.63}{0.0007246} = 2249.4$$

The b and δb were chosen to give the maximum, so $\kappa(A) = 2249.4$

Condition Number and Gaussian Elimination

The **Condition Number** plays a fundamental role in the analysis of ***roundoff errors*** during the solution of **Gaussian Elimination**

Assume that A and b are exact and x_* is obtained from the floating point arithmetic of the machine with ϵ precision

Assuming no singularities, the following inequalities can be established:

$$\begin{aligned} \frac{\|b - Ax_*\|}{\|A\|\|x_*\|} &\leq \rho\epsilon, \\ \frac{\|x - x_*\|}{\|x_*\|} &\leq \rho\kappa(A)\epsilon, \end{aligned}$$

where $\rho \approx 10$

The first inequality implies that the ***relative residual*** is approximately the same as the machine ***roundoff error***

The second inequality implies that the ***relative error*** requires A is nonsingular, and the ***roundoff error*** scales with the **condition number,** $\kappa(A)$