# Math 541 - Numerical Analysis

## Lecture Notes – Computer Arithmetic and Finite Precision

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## Outline







## Finite Precision — A single `char`

Computers use a finite number of bits (0's and 1's) to represent numbers.

For instance, an 8-bit unsigned integer (a.k.a a "char") is stored:

| $2^7$ | $2^6$ | $2^5$ | $2^4$ | $2^3$ | $2^2$ | $2^1$ | $2^0$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |

Here, $2^6 + 2^3 + 2^2 + 2^0 = 64 + 8 + 4 + 1 = 77$, which represents the upper-case character "M" (US-ASCII).





## Finite Precision — A 64-bit real number, `double`

The ***Binary Floating Point Arithmetic Standard* 754-1985** (IEEE — The Institute for Electrical and Electronics Engineers) standard specified the following layout for a 64-bit real number:

$$\mathbf{s\, c_{10}\, c_9\, \ldots\, c_1\, c_0\, m_{51}\, m_{50}\, \ldots\, m_1\, m_0}$$

where

| Symbol | Bits | Description |
|---|---|---|
| $s$ | 1 | The sign bit — 0=positive, 1=negative |
| $c$ | 11 | The characteristic (exponent) |
| $m$ | 52 | The mantissa |

$$r = (-1)^s\, 2^{c-1023}\,(1+m), \quad c = \sum_{k=0}^{10} c_k 2^k, \quad m = \sum_{k=0}^{51} \frac{m_k}{2^{52-k}}$$

## Examples: Finite Precision

$$r = (-1)^s\, 2^{c-1023}\,(1+f), \quad c = \sum_{k=0}^{10} c_k 2^k, \quad m = \sum_{k=0}^{51} \frac{m_k}{2^{52-k}}$$

**Example 1:** The number **3.0**

0 10000000000 1000000000000000000000000000000000000000000000000000

$$r_1 = (-1)^0 \cdot 2^{2^{10}-1023} \cdot \left(1 + \frac{1}{2}\right) = 1 \cdot 2^1 \cdot \frac{3}{2} = 3.0$$

**Example 2:** The **Smallest Positive Real Number**

0 00000000000 0000000000000000000000000000000000000000000000000001

$$r_2 = (-1)^0 \cdot 2^{0-1023} \cdot (1 + 2^{-52}) = (1 + 2^{-52}) \cdot 2^{-1023} \cdot 1 \approx 10^{-308}$$





## Examples: Finite Precision

$$r = (-1)^s\, 2^{c-1023}\,(1+f), \quad c = \sum_{k=0}^{10} c_k 2^k, \quad m = \sum_{k=0}^{51} \frac{m_k}{2^{52-k}}$$

**Example 3:** The **Largest Positive Real Number**

0 11111111110 1111111111111111111111111111111111111111111111111111

$$\begin{aligned} r_3 &= (-1)^0 \cdot 2^{1023} \cdot \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{51}} + \frac{1}{2^{52}}\right) \\ &= 2^{1024} \cdot \left(2 - \frac{1}{2^{52}}\right) \approx 10^{308} \end{aligned}$$





## Special Numbers

Note that the IEEE standard does NOT allow **zero!**

- There are some special signals in IEEE-754-1985:
- All zeros for $c$ and $m$ produce **zero**
- $c$ having 11 bits all 1 gives either $NaN$ (Not a Number) or $\pm\infty$
- Further reference at http://www.freesoft.org/CIE/RFC/1832/32.htm





## Something is Missing — Gaps in the Representation



***There are gaps in the floating-point representation!***

Given the representation

0 00000000000 00000000000000000000000000000000000000000000000000**01**

for the value $\frac{2^{-1023}}{2^{52}}$.

The next larger floating-point value is

0 00000000000 00000000000000000000000000000000000000000000000000**10**

*i.e.* the value $\frac{2^{-1023}}{2^{51}}$.

The difference between these two values is $\frac{2^{-1023}}{2^{52}} = 2^{-1075}$, so any number in the interval $\left(\frac{2^{-1023}}{2^{52}}, \frac{2^{-1023}}{2^{51}}\right)$ **is not representable!**

## Something is Missing — Gaps in the Representation 

A gap of $2^{-1075}$ doesn't seem too bad...

However, the size of the gap depends on the value itself...

Consider $r = 3.0$

0 10000000000 1000000000000000000000000000000000000000000000000000

and the next value

0 10000000000 1000000000000000000000000000000000000000000000000001

The difference is $\frac{2}{2^{52}} \approx 4.4 \cdot 10^{-16}$





## Something is Missing — Gaps in the Representation 

At the other extreme, the difference between

0 11111111110 1111111111111111111111111111111111111111111111111111

and the previous value

0 11111111110 1111111111111111111111111111111111111111111111111110

is $\frac{2^{1023}}{2^{52}} = 2^{971} \approx 1.99 \cdot 10^{292}$.

That's a "fairly significant" gap!!!

***The number of atoms in the observable universe can be estimated to be no more than $\sim 10^{80}$.***





## The Relative Gap

It makes more sense to factor the exponent out of the discussion and talk about the relative gap:

| Exponent | Gap | Relative Gap (Gap/Exponent) |
|---|---|---|
| $2^{-1023}$ | $2^{-1075}$ | $2^{-52}$ |
| $2^{1}$ | $2^{-51}$ | $2^{-52}$ |
| $2^{1023}$ | $2^{971}$ | $2^{-52}$ |

Any difference between numbers smaller than the local gap is not representable, *e.g.* any number in the interval

$$\left[3.0,\ 3.0 + \tfrac{1}{2^{51}}\right)$$

is represented by the value 3.0.

The **MatLab** command `eps` (for epsilon tolerance) gives ***double precision***, which is

$$2^{-52} \approx 2.2204 \times 10^{-16}$$





## The Floating Point "Numbers"

***Floating point "numbers" represent intervals!***

Since (most) humans find it hard to think in binary representation, from now on we will **for simplicity** and **without loss of generality** assume that floating point numbers are represented in the normalized floating point form as...

**$k$-digit decimal machine numbers**

$$\pm 0.d_1 d_2 \cdots d_{k-1} d_k \cdot 10^n$$

where

$$1 \le d_1 \le 9, \quad 0 \le d_i \le 9,\ i \ge 2, \quad n \in \mathbb{Z}$$

## $k$-Digit Decimal Machine Numbers

Any real number can be written in the form

$$r = \pm 0.d_1 d_2 \cdots d_\infty \cdot 10^n$$

given infinite patience and storage space.

We can obtain the floating-point representation `fl(r)` in two ways:

1. Truncating (chopping) — just keep the first $k$ digits (In **MatLab** use `floor(r)`)
2. Rounding — if $d_{k+1} \geq 5$ then add 1 to $d_k$. Truncate. (Standard for most languages)

**Examples**

$$\mathtt{fl}_{t,5}(\pi) = 0.31415 \cdot 10^1, \quad \mathtt{fl}_{r,5}(\pi) = 0.31416 \cdot 10^1$$

In both cases, the error introduced is called the **roundoff error**.





## Quantifying the Error

Let $p^*$ be and approximation to $p$, then...

**Definition (The Absolute Error)**

$$|p - p^*|$$

**Definition (The Relative Error)**

$$\frac{|p - p^*|}{|p|}, \quad p \neq 0$$

**Definition (Significant Digits)**

The number of **significant digits** is the largest value of $t$ for which

$$\frac{|p - p^*|}{|p|} < 5 \cdot 10^{-t}$$





## Sources of Numerical Error *Important!!!*

**Some Sources of Numerical Error**

1. **Representation — Roundoff**
2. **Cancellation**

Consider:

$$\begin{array}{rl} & 0.12345678012345 \cdot 10^1 \\ - & 0.12345678012344 \cdot 10^1 \\ \hline = & 0.10000000000000 \cdot 10^{-13} \end{array}$$

this value has (at most) **1** significant digit!!!

If you assume a "cancelled value" has more significant bits (the computer will happily give you some numbers) — Any use of these random digits is **GARBAGE**!!!





## Examples: 5-digit Arithmetic $k$-Digit Decimal Machine Numbers

**Rounding 5-digit arithmetic**

$$\begin{gathered}(0.96384 \cdot 10^5 + 0.26678 \cdot 10^2) - 0.96410 \cdot 10^5 = \\ (0.96384 \cdot 10^5 + 0.00027 \cdot 10^5) - 0.96410 \cdot 10^5 = \\ 0.96411 \cdot 10^5 - 0.96410 \cdot 10^5 = 0.10000 \cdot 10^1\end{gathered}$$

**Truncating 5-digit arithmetic**

$$\begin{gathered}(0.96384 \cdot 10^5 + 0.26678 \cdot 10^2) - 0.96410 \cdot 10^5 = \\ (0.96384 \cdot 10^5 + 0.00026 \cdot 10^5) - 0.96410 \cdot 10^5 = \\ 0.96410 \cdot 10^5 - 0.96410 \cdot 10^5 = 0.0000 \cdot 10^0\end{gathered}$$

**Rearrangement changes the result:**

$$\begin{gathered}(0.96384 \cdot 10^5 - 0.96410 \cdot 10^5) + 0.26678 \cdot 10^2 = \\ -0.26000 \cdot 10^2 + 0.26678 \cdot 10^2 = 0.67800 \cdot 10^0\end{gathered}$$

**Numerically, order of computation matters!** (This is a **HARD** problem)

## Example: Loss of Significant Digits due to Subtractive Cancellation

Consider the recursive relation

$$x_{n+1} = 1 - (n+1)x_n \quad \text{with} \quad x_0 = 1 - \frac{1}{e}.$$

This sequence can be shown to converge to **0**

**Subtractive cancellation** produces an error, which is approximately equal to the machine precision times $n!$.





## Example: Proof of Convergence to 0

The ***recursive relation*** is

$$x_{n+1} = 1 - (n+1)x_n$$

with

$$x_0 = 1 - \frac{1}{e} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots$$

From the recursive relation

$$\begin{aligned}
x_1 &= 1 - x_0 = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots \\
x_2 &= 1 - 2x_1 = \frac{1}{3} - \frac{2}{4!} + \frac{2}{5!} - \dots \\
x_3 &= 1 - 3x_2 = \frac{3!}{4!} - \frac{3!}{5!} + \frac{3!}{6!} - \dots \\
&\vdots \\
x_n &= 1 - nx_{n-1} = \frac{n!}{(n+1)!} - \frac{n!}{(n+2)!} + \frac{n!}{(n+3)!} - \dots
\end{aligned}$$

This shows that

$$x_n = \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \dots \to 0 \quad \text{as} \quad n \to \infty.$$





## Subtraction Error

The ***recursive relation*** $x_{n+1} = 1 - (n+1)x_n \quad \text{with} \quad x_0 = 1 - \frac{1}{e}$

```
clear
x(1) = 1-1/exp(1);
s(1) = 1;
f(1) = 1;
for i = 2:21
x(i) = 1-(i-1)*x(i-1);
s(i) = 1/i;
f(i) = (i-1)*f(i-1);
end
n = 0:20;
z = [n; x; s; f];
fprintf(1, '\n\n n x(n) 1/(n+1) n!\n\n')
fprintf(1, '%2.0f %13.8f %10.8f %10.3g\n',z)
```





## Subtractive Cancellation Example: Output

| $n$ | $x_n$ | $n!$ | $n$ | $x_n$ | $n!$ |
|---|---|---|---|---|---|
| 0 | 0.63212056 | 1 | 11 | 0.07735223 | 3.99e+007 |
| 1 | 0.36787944 | 1 | 12 | 0.07177325 | 4.79e+008 |
| 2 | 0.26424112 | 2 | 13 | 0.06694778 | 6.23e+009 |
| 3 | 0.20727665 | 6 | 14 | 0.06273108 | 8.72e+010 |
| 4 | 0.17089341 | 24 | 15 | 0.05903379 | 1.31e+012 |
| 5 | 0.14553294 | 120 | 16 | 0.05545930 | 2.09e+013 |
| 6 | 0.12680236 | 720 | 17 | 0.05719187 | 3.56e+014 |
| 7 | 0.11238350 | 5.04e+003 | 18 | −0.02945367 | 6.4e+015 |
| 8 | 0.10093197 | 4.03e+004 | 19 | 1.55961974 | 1.22e+017 |
| 9 | 0.09161229 | 3.63e+005 | 20 | −30.19239489 | 2.43e+018 |
| 10 | 0.08387707 | 3.63e+006 | | | |

# Subtraction Error

Consider the **MatLab** computation near $x = 1$ of

$$y = x^7 - 7\,x^6 + 21\,x^5 - 35\,x^4 + 35\,x^3 - 21\,x^2 + 7\,x - 1$$

compared to $\quad y = (x-1)^7$

```
% Rounding Error Graph

x = 0.988:0.0001:1.012;
y = x.^7 - 7*x.^6 + 21*x.^5 - 35*x.^4 + ...
    35*x.^3 - 21*x.^2 + 7*x - 1;
yy = (x - 1).^7;

plot(x,y,'k-','linewidth',1.5);
hold on
plot(x,yy,'r-','linewidth',1.5);
grid
```

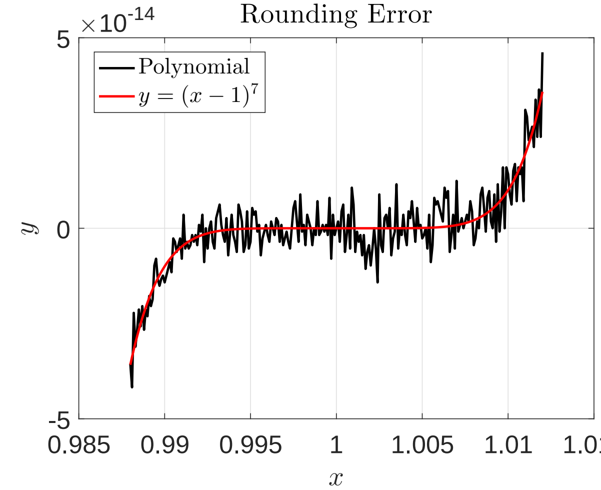




# Subtraction Error

The program graphs $x \in [0.988, 1.012]$ with the two forms of function:

$$y = x^7 - 7\,x^6 + 21\,x^5 - 35\,x^4 + 35\,x^3 - 21\,x^2 + 7\,x - 1 = (x-1)^7$$








# Algorithms

**Definition (Algorithm)**

An **algorithm** is a procedure that describes, in an ***unambiguous manner***, a finite sequence of steps to be performed in a specific order.

In this class, the objective of an algorithm is to implement a procedure to solve a problem or approximate a solution to a problem.

There are many collection of algorithms "out there" called ***Numerical Recipes***





# Key Concepts for Numerical Algorithms Stability

**Definition (Stability )**

An algorithm is said to be stable if small changes in the input, generates small changes in the output.

- At some point we need to quantify what "small" means!
- If an algorithm is stable for a certain ***range*** of initial data, then is it said to be ***conditionally stable***.
- Stability issues are discussed in great detail in ***Math 543*** and our **Dynamical Systems** classes.

## Key Concepts for Numerical Algorithms — Error Growth

Suppose $E_0 > 0$ denotes the initial error, and $E_n$ represents the error after $n$ operations.

- If $E_n \approx \mathcal{C} E_0 \cdot n$ (for a constant $\mathcal{C}$, which is independent of $n$), then the growth is ***linear***.
- If $E_n \approx \mathcal{C}^n E_0$, $\mathcal{C} > 1$, then the growth is ***exponential*** — in this case the error will dominate very fast(undesirable scenario).
- **Linear error growth** is usually unavoidable, and in the case where $\mathcal{C}$ and $E_0$ are small the results are generally acceptable. — **Stable algorithm**.
- **Exponential error growth** is unacceptable. Regardless of the size of $E_0$ the error grows rapidly. — **Unstable algorithm**.
- One property of **chaos** in a dynamical system is the ***exponential growth*** of any error in initial conditions – leading to **unpredictable behavior**





## Example — 1 of 2

The ***recursive equation***

$$p_n = \frac{10}{3} p_{n-1} - p_{n-2}, \quad n = 2, 3, \ldots, \infty$$

has the **exact solution**

$$p_n = c_1 \left(\frac{1}{3}\right)^n + c_2 3^n$$

for any constants $c_1$ and $c_2$. (Determined by starting values.)

In particular, if $p_0 = 1$ and $p_1 = \frac{1}{3}$, we get $c_1 = 1$ and $c_2 = 0$, so $p_n = \left(\frac{1}{3}\right)^n$ for all $n$.

What happens with some rounding error, as we don't know $\frac{1}{3}$ exactly?





## Example — 2 of 2

Consider what happens in ***5-digit rounding arithmetic***, where the initial starting conditions are rounded.

$$p_0^* = 1.0000, \quad p_1^* = 0.33333$$

which modifies the constants (by solving the general solution for $c_1$ and $c_2$ with the $p_0^*$ and $p_1^*$)

$$c_1^* = 1.0000, \quad c_2^* = -0.12500 \cdot 10^{-5}$$

The generated sequence is

$$p_n^* = 1.0000\,(0.33333)^n - \underbrace{0.12500 \cdot 10^{-5}(3.0000)^n}_{\textbf{Exponential Growth}}$$

$p_n^*$ quickly becomes a very poor approximation to $p_n$ due to the ***exponential growth*** of the initial roundoff error.





## Reducing the Effects of Roundoff Error

- The effects of roundoff error can be reduced by using higher-order-digit arithmetic such as the **double** or **multiple-precision arithmetic** available on most computers.
- Disadvantages in using **double precision arithmetic** are that it takes more computation time, and ***the growth of the roundoff error is not eliminated but only postponed.***
- Sometimes, but not always, it is possible to reduce the growth of the roundoff error by restructuring the calculations.

## Key Concepts — Rate of Convergence

**Definition (Rate of Convergence)**

Suppose the sequence $\underline{\beta} = \{\beta_n\}_{n=1}^{\infty}$ converges to zero, and $\underline{\alpha} = \{\alpha_n\}_{n=1}^{\infty}$ converges to a number $\alpha$.

If there exists $K > 0$: $|\alpha_n - \alpha| < K\beta_n$, for $n$ large enough, then we say that $\{\alpha_n\}_{n=1}^{\infty}$ converges to $\alpha$ with a **Rate of Convergence** $\mathcal{O}(\beta_n)$ ("Big Oh of $\beta_n$").

We write

$$\alpha_n = \alpha + \mathcal{O}(\beta_n)$$

**Note:** The sequence $\underline{\beta} = \{\beta_n\}_{n=1}^{\infty}$ is usually chosen to be

$$\beta_n = \frac{1}{n^p}$$

for some positive value of $p$.





## Examples: Rate of Convergence

**Example 1:**
If

$$\alpha_n = \alpha + \frac{1}{\sqrt{n}}$$

then for any $\varepsilon > 0$

$$|\alpha_n - \alpha| = \frac{1}{\sqrt{n}} \leq \underbrace{(1+\varepsilon)}_{K}\underbrace{\frac{1}{\sqrt{n}}}_{\beta_n}$$

Hence,

$$\alpha_n = \alpha + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$$





## Examples: Rate of Convergence

**Example 2:** Consider the sequence (as $n \to \infty$)

$$\alpha_n = \sin\left(\frac{1}{n}\right) - \frac{1}{n}$$

The **Maclaurin series** expansion for $\sin(x)$ is:

$$\sin\left(\frac{1}{n}\right) \sim \frac{1}{n} - \frac{1}{6n^3} + \mathcal{O}\left(\frac{1}{n^5}\right)$$

Hence

$$|\alpha_n| = \left|\frac{1}{6n^3} + \mathcal{O}\left(\frac{1}{n^5}\right)\right|$$

It follows that

$$\alpha_n = \mathbf{0} + \mathcal{O}\left(\frac{1}{n^3}\right)$$

Note:

$$\mathcal{O}\left(\frac{1}{n^3}\right) + \mathcal{O}\left(\frac{1}{n^5}\right) = \mathcal{O}\left(\frac{1}{n^3}\right), \quad \text{since} \quad \frac{1}{n^5} \ll \frac{1}{n^3}, \quad \text{as} \quad n \to \infty$$





## Generalizing to Continuous Limits

**Definition (Rate of Convergence)**

Suppose

$$\lim_{h\to 0} G(h) = 0, \quad \text{and} \quad \lim_{h\to 0} F(h) = L$$

If there exists $K > 0$:

$$|F(h) - L| \leq K\,|G(h)|$$

for all $h < H$ (for some $H > 0$), then

$$F(h) = L + \mathcal{O}(G(h))$$

we say that $F(h)$ converges to $L$ with a **Rate of Convergence** $\mathcal{O}(G(h))$.

Usually $G(h) = h^p$, $p > 0$.

# Examples: Rate of Convergence

**Example 2-b:** Consider the function $\alpha(h)$ (as $h \to 0$)

$$\alpha(h) = \sin(h) - h$$

The **Maclaurin series** expansion for $\sin(x)$ is:

$$\sin(h) \sim h - \frac{h^3}{6} + \mathcal{O}\left(h^5\right)$$

Hence

$$|\alpha(h)| = \left|\frac{h^3}{6} + \mathcal{O}\left(h^5\right)\right|$$

It follows that

$$\lim_{h \to 0} \alpha(h) = \mathbf{0} + \mathcal{O}\left(h^3\right)$$

Note:

$$\mathcal{O}\left(h^3\right) + \mathcal{O}\left(h^5\right) = \mathcal{O}\left(h^3\right), \quad \text{since} \quad h^5 \ll h^3, \quad \text{as} \quad h \to 0$$