1. a. For $\dot{\mathbf{x}} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}$, the characteristic equation satisfies

$$\det\begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2) = 0,$$

so $\lambda_1 = 2$. With this eigenvalue, the eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. The other eigenvalue is $\lambda_2 = 4$, and its associated eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. It follows that the general solution is

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t},$$

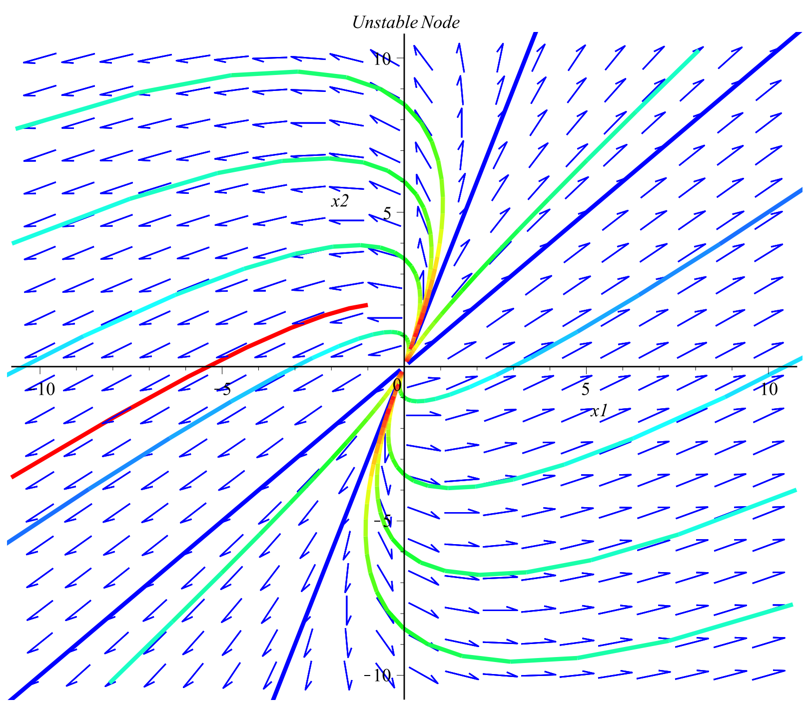
which is an **unstable node**. To satisfy the initial value problem, solve

$$\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \text{so} \quad c_1 = 1.5 \quad \text{and} \quad c_2 = -2.5.$$

This gives the unique solution

$$\mathbf{x}(t) = \begin{pmatrix} 1.5 \\ 4.5 \end{pmatrix}e^{2t} - \begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix}e^{4t}.$$

Below shows the phase portrait with the eigenvectors (blue), the solution to the IVP (red), and typical solutions (rainbow).



b. For $\dot{\mathbf{x}} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$, the characteristic equation satisfies

$$\det \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 1 = 0,$$

so $\lambda_{1,2} = \pm i$. For eigenvalue $\lambda_1 = i$, the eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$. The complex solution is

$$\mathbf{x}_1(t) = \begin{pmatrix} 5 \\ 2-i \end{pmatrix} (\cos(t) + i\sin(t)) = \begin{pmatrix} 5\cos(t) + 5i\sin(t) \\ 2\cos(t) + \sin(t) + i(2\sin(t) - \cos(t)) \end{pmatrix}.$$

Taking the real and imaginary parts gives the general solution

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix} + c_2 \begin{pmatrix} 5\sin(t) \\ 2\sin(t) - \cos(t) \end{pmatrix},$$

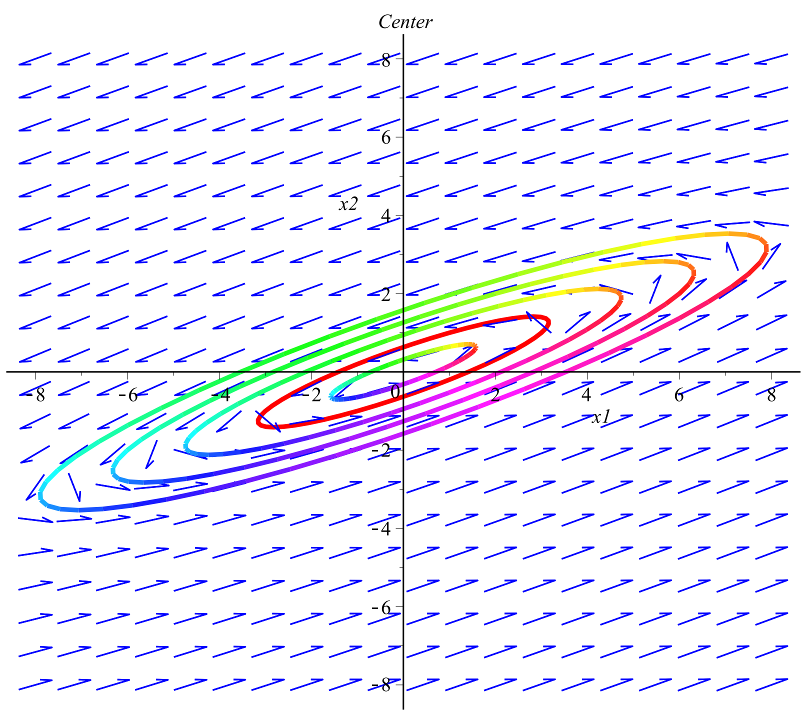
which is a **center**. To satisfy the initial value problem, solve

$$\begin{pmatrix} 5 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \text{so} \quad c_1 = \frac{3}{5} \quad \text{and} \quad c_2 = \frac{1}{5}.$$

This gives the unique solution

$$\mathbf{x}(t) = \frac{3}{5} \begin{pmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 5\sin(t) \\ 2\sin(t) - \cos(t) \end{pmatrix},$$

Below shows the phase portrait with the solution to the IVP (red) and typical trajectories (rainbow), where the trajectories travel counterclockwise.



c. For $\dot{\mathbf{x}} = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \mathbf{x}$, the characteristic equation satisfies

$$\det \begin{vmatrix} -1-\lambda & 2 \\ 2 & -4-\lambda \end{vmatrix} = \lambda^2 + 5\lambda = \lambda(\lambda+5) = 0,$$

so $\lambda_1 = -5$. With this eigenvalue, the eigenvector is $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. The other eigenvalue is $\lambda_2 = 0$, and its associated eigenvector $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. It follows that the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

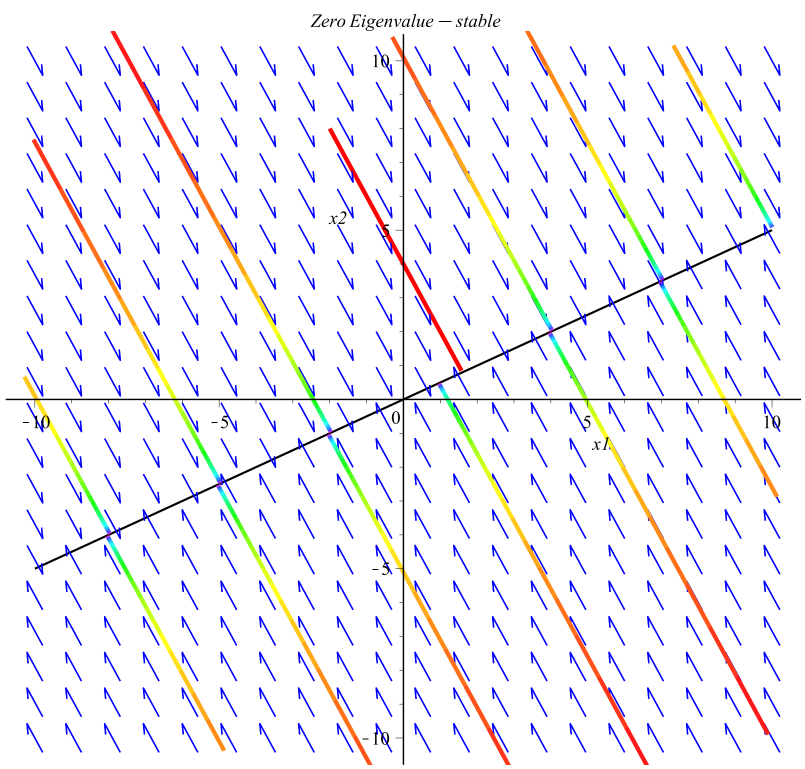
which has a **line of equilibria** ($\lambda_2 = 0$), which are **stable**. To satisfy the initial value problem, solve

$$\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \end{pmatrix}, \quad \text{so} \quad c_1 = 3.6 \quad \text{and} \quad c_2 = 0.8.$$

This gives the unique solution

$$\mathbf{x}(t) = 3.6 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-5t} + 0.8 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Below shows the phase portrait with the line of equilibria (black line), the solution to the IVP (red), and typical trajectories (rainbow).



d. For $\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -8 & -4 \end{pmatrix} \mathbf{x}$, the characteristic equation satisfies

$$\det \begin{vmatrix} -\lambda & 1 \\ -8 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 8 = 0,$$

so $\lambda_{1,2} = -2 \pm 2i$. For eigenvalue $\lambda_1 = -2+2i$, the eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2+2i \end{pmatrix}$. The complex solution is

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ -2+2i \end{pmatrix} e^{-2t}(\cos(2t)+i\sin(2t)) = e^{-2t} \begin{pmatrix} (\cos(2t)+i\sin(2t)) \\ -2\cos(t) - 2\sin(t) + i(-2\sin(t) + 2\cos(t)) \end{pmatrix}.$$

Taking the real and imaginary parts gives the general solution

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{pmatrix} \cos(t) \\ -2\cos(t) - 2\sin(t) \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} \sin(t) \\ -2\sin(t) + 2\cos(t) \end{pmatrix},$$

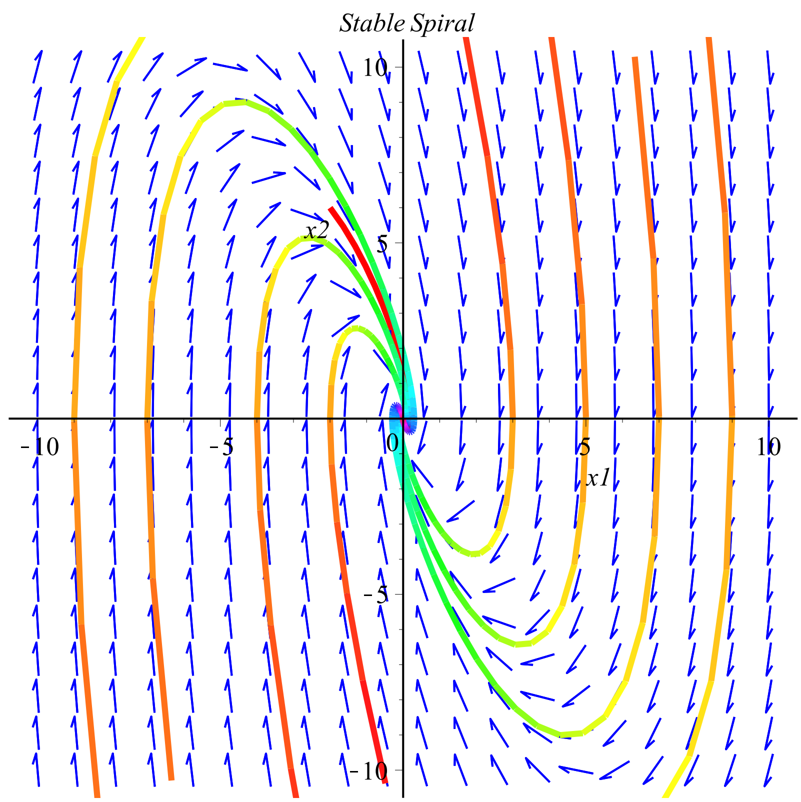
which is a **stable spiral**. To satisfy the initial value problem, solve

$$\begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}, \quad \text{so} \quad c_1 = -2 \quad \text{and} \quad c_2 = 1.$$

This gives the unique solution

$$\mathbf{x}(t) = -2e^{-2t} \begin{pmatrix} \cos(t) \\ -2\cos(t) - 2\sin(t) \end{pmatrix} + e^{-2t} \begin{pmatrix} \sin(t) \\ -2\sin(t) + 2\cos(t) \end{pmatrix},$$

Below shows the phase portrait with the solution to the IVP (red) and typical trajectories (rainbow), where the trajectories travels clockwise.



e. For $\dot{\mathbf{x}} = \begin{pmatrix} -1 & -\frac{1}{2} \\ 2 & -3 \end{pmatrix} \mathbf{x}$, the characteristic equation satisfies

$$\det \begin{vmatrix} -1-\lambda & -0.5 \\ 2 & -3-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = 0,$$

so $\lambda = -2$ is a repeated eigenvalue with an eigenspace of dimension 1 spanned by the eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The second solution requires solving $(\mathbf{A} + 2\mathbf{I})\mathbf{w} = \mathbf{v}$ (for the higher null space) or

$$\begin{pmatrix} 1 & -0.5 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which has the solution $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. It follows that the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{-2t},$$

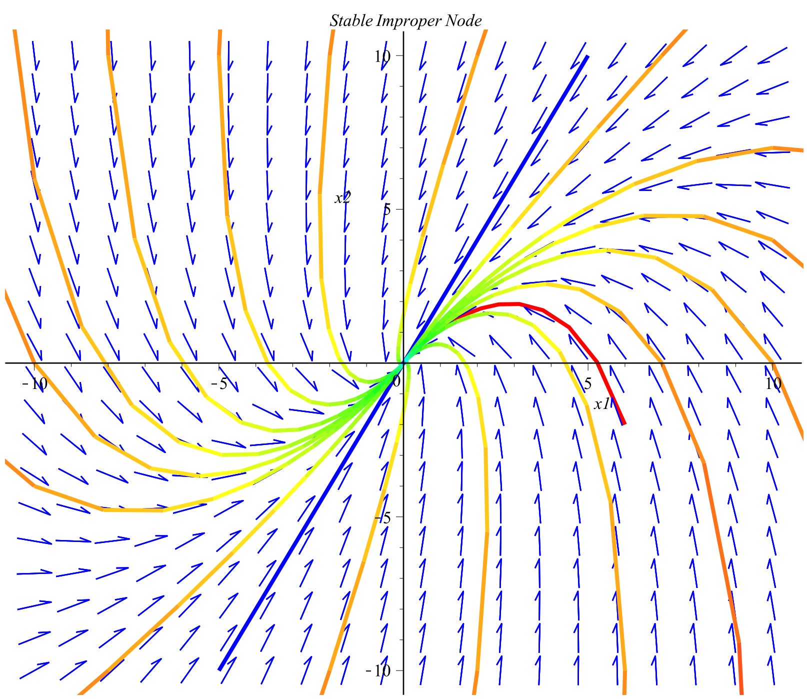
which is a **stable improper node**. To satisfy the initial value problem, solve

$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}, \quad \text{so} \quad c_1 = -1 \quad \text{and} \quad c_2 = 7.$$

This gives the unique solution

$$\mathbf{x}(t) = -\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + 7 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{-2t}.$$

Below shows the phase portrait with the one eigenvector (blue), the solution to the IVP (red), and typical trajectories (rainbow).



f. For $\dot{\mathbf{x}} = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}$, the characteristic equation satisfies

$$\det \begin{vmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3) = 0,$$

so $\lambda_1 = -1$. With this eigenvalue, the eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The other eigenvalue is $\lambda_2 = 3$, and its associated eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$. It follows that the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t},$$

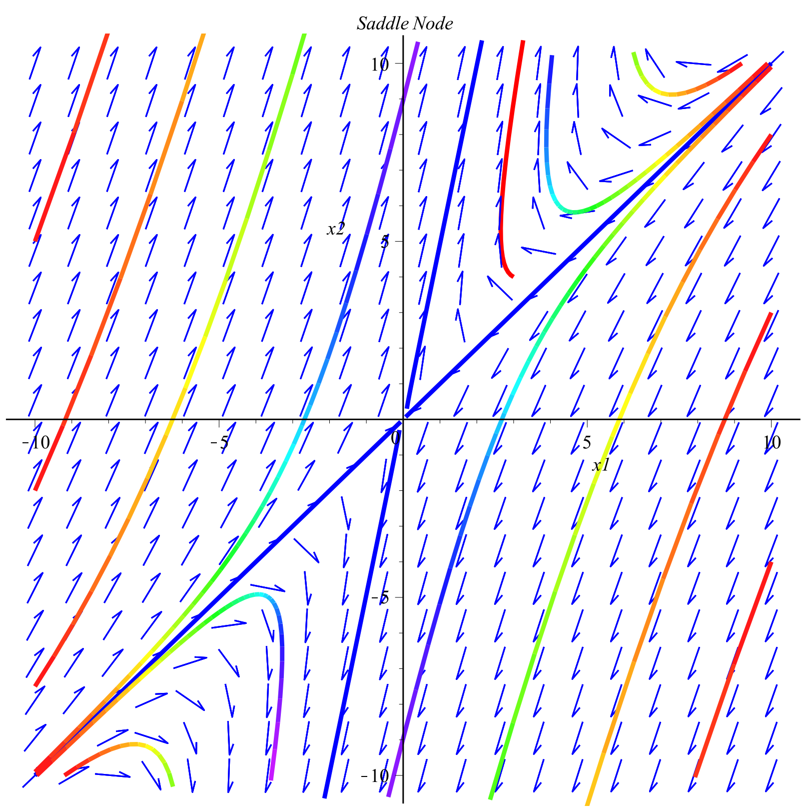
which is a **saddle node**. To satisfy the initial value problem, solve

$$\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \text{so} \quad c_1 = 2.75 \quad \text{and} \quad c_2 = 0.25.$$

This gives the unique solution

$$\mathbf{x}(t) = 2.75 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + 0.25 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}.$$

Below shows the phase portrait with the eigenvectors (blue), the solution to the IVP (red), and typical solutions (rainbow).



2. The system given by:

$$\dot{\mathbf{x}} = \begin{pmatrix} -1 & \alpha \\ -1 & -1 \end{pmatrix} \mathbf{x}$$

has the characteristic equation

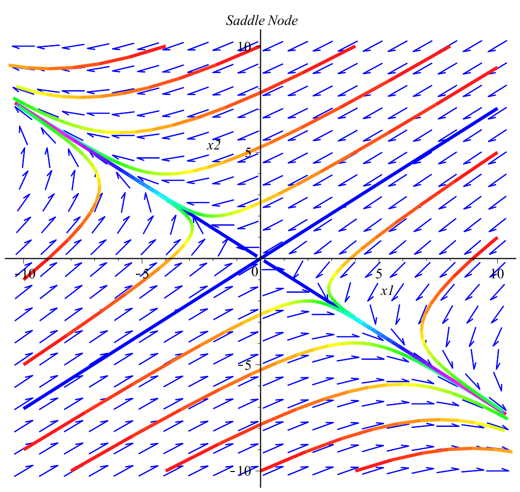
$$\det \begin{vmatrix} -1-\lambda & \alpha \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 + \alpha = 0.$$

This has the eigenvalues $\lambda = -1 \pm \sqrt{-\alpha}$. There are clearly qualitative changes at $\alpha = -1$ and $\alpha = 0$.

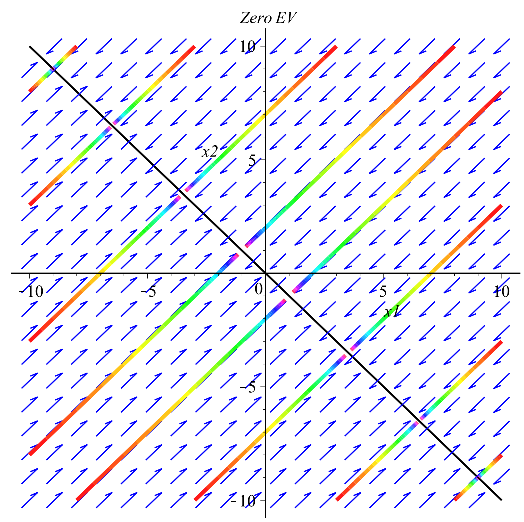
At $\alpha = -1$, the system has the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 0$. The latter, $\lambda_2 = 0$, leads to the degenerate case where the system has a **line of equilibria**. The general solution satisfies:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}.$$

For $\alpha < -1$, the system has a **saddle node** (a positive and a negative eigenvalue) with a typical phase portrait ($\alpha = -2$) as shown below on the left. When $\alpha = -1$, there is the degenerate case with a **line of equilibria** and all solutions converging to that line ($x_1 = -x_2$). This is shown in the phase portrait below on the right

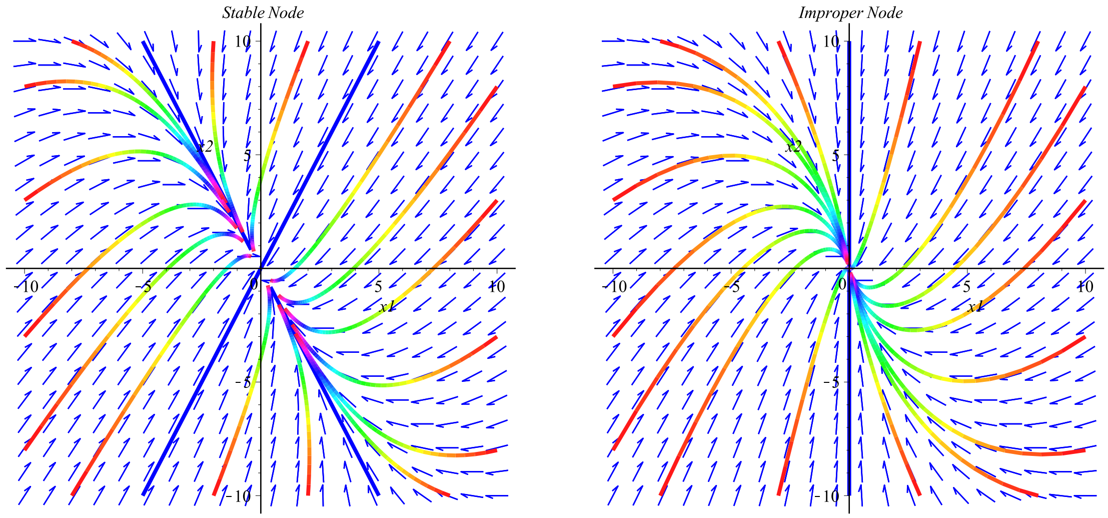


$\alpha < -1$, **Saddle Node**



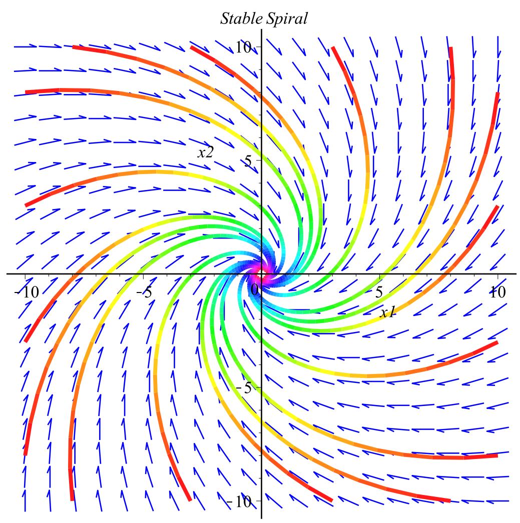
$\alpha = -1$, **Line of Equilibria**

For $-1 < \alpha < 0$, the system has a **stable node** ($\lambda_1 < \lambda_2 < 0$) with a typical phase portrait ($\alpha = -0.25$) as shown below on the left. When $\alpha = 0$, there is a **stable improper node** with both eigenvalues, $\lambda = -1$. This is shown in the phase portrait on the right.



$-1 < \alpha < 0$, **Stable Node** $\alpha = 0$, **Stable Improper Node**

Finally, for $\alpha > 0$, the eigenvalues have complex values with negative real parts, which results in a **stable spiral**. The phase portrait is below.



$\alpha > 0$, **Stable Spiral**

3. First, find the equilibrium for:

$$\begin{aligned} \frac{dx_1}{dt} &= -x_1 - 4x_2 + 6, \\ \frac{dx_2}{dt} &= x_1 - x_2 + 4, \end{aligned}$$

so solve

$$\begin{aligned} x_{1e} + 4x_{2e} &= 6, \\ x_{1e} - x_{2e} &= -4. \end{aligned}$$

Thus, $5x_{2e} = 10$ or $x_{2e} = 2$ and $x_{1e} = -2$.

Make the change of variables, $z_1(t) = x_1(t) + 2$ and $z_2(t) = x_2(t) - 2$. The DE for $\mathbf{z}(t)$ is

$$\dot{\mathbf{z}} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{z}.$$

This has the characteristic equation

$$\det \begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 5 = (\lambda+1)^2 + 4 = 0.$$

It follows that the eigenvalues are $\lambda_{1,2} = -1 \pm 2i$. For the eigenvalue $\lambda_1 = -1 + 2i$, there is the eigenvector $\mathbf{v}_1 = \begin{pmatrix} 4 \\ -2i \end{pmatrix}$, so the complex solution is

$$\mathbf{z}_1(t) = \begin{pmatrix} 4 \\ -2i \end{pmatrix} e^{-t}(\cos(2t) + i\sin(2t)) = e^{-t} \begin{pmatrix} 4\cos(2t) + 4i\sin(2t) \\ 2\sin(2t) - 2i\cos(2t) \end{pmatrix}.$$

The general solution for this problem is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 4\cos(2t) \\ 2\sin(2t) \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 4\sin(2t) \\ -2\cos(2t) \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

We solve the IVP, so

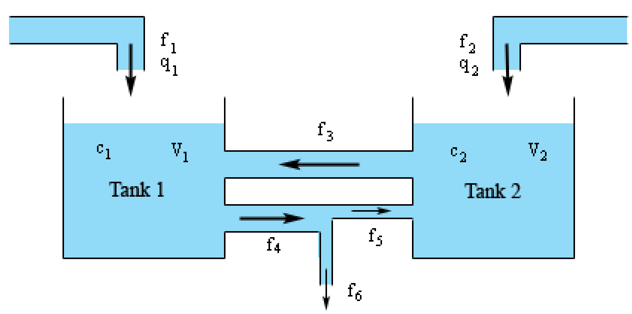
$$\mathbf{x}(0) = c_1 \begin{pmatrix} 4 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 4c_1 \\ -2c_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

It follows that $c_1 = \frac{7}{4}$ and $c_2 = 0$, so

$$\mathbf{x}(t) = \frac{7e^{-t}}{4} \begin{pmatrix} 4\cos(2t) \\ 2\sin(2t) \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

4. From the information, we write the rate of change in **amounts**, $A_1(t)$ and $A_2(t)$. The rate of change in amount is concentration times flow rate.

$$\frac{dA_i}{dt} = \textit{amount enter} - \textit{amount leave}.$$



For Tank 1, the amount entering is $f_1 q_1$ and $f_3 c_2$, while the amount leaving is $f_4 c_1$. Similar expressions give the equation for Tank 2. The concentration equations follow by simply dividing the amount equations by the appropriate volumes.

With the data the amount equations are

$$\begin{aligned} \frac{dA_1}{dt} &= 0.3 \cdot 8 + 0.4 \cdot c_2 - 0.7 \cdot c_1, \\ \frac{dA_2}{dt} &= 0.2 \cdot 15 + 0.2 \cdot c_1 - 0.4 \cdot c_2. \end{aligned}$$

Dividing by the volumes gives the concentration equations

$$\begin{aligned}\frac{dc_1}{dt} &= -\frac{7}{2000}c_1 + \frac{1}{500}c_2 + \frac{3}{250},\\ \frac{dc_2}{dt} &= \frac{1}{500}c_1 - \frac{1}{250}c_2 + \frac{3}{100}.\end{aligned}$$

The equilibria are found first by solving:

$$\begin{aligned}\frac{7}{2000}c_{1e} - \frac{1}{500}c_{2e} &= \frac{3}{250},\\ -\frac{1}{500}c_{1e} + \frac{1}{250}c_{2e} &= \frac{3}{100}.\end{aligned}$$

It follows that $c_{1e} = 10.8$ g/l, while $c_{2e} = 12.9$ g/l. This gives the equilibrium solution, which will be the asymptotic limit for the concentrations.

We make a change of variables $z_1(t) = c_1(t) - 10.8$ and $z_2(t) = c_2(t) - 12.9$. The homogeneous equation, $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$, is

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} -0.0035 & 0.002 \\ 0.002 & -0.004 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

The characteristic equation is

$$\det \begin{vmatrix} -0.0035 - \lambda & 0.002 \\ 0.002 & -0.004 - \lambda \end{vmatrix} = \lambda^2 + 0.0075\lambda + 0.00001 = 0,$$

which gives $\lambda_1 = -0.0057656$ with its associated eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1.13278 \end{pmatrix}$ and $\lambda_2 = -0.0017344$ with its associated eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1.13278 \\ 1 \end{pmatrix}$. This results in the general solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1.13278 \end{pmatrix} e^{-0.0057656t} + c_2 \begin{pmatrix} 1.13278 \\ 1 \end{pmatrix} e^{-0.0017344t} + \begin{pmatrix} 10.8 \\ 12.9 \end{pmatrix},$$

which is a **stable node**. The solution to the IVP satisfies

$$\begin{pmatrix} 1 & 1.13278 \\ -1.13278 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 - 10.8 \\ 3 - 12.9 \end{pmatrix} = \begin{pmatrix} -8.8 \\ -9.9 \end{pmatrix},$$

which gives $c_1 = 1.05752$ and $c_2 = -8.70206$. It follows that the unique solution to the IVP is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1.05752 \\ -1.19794 \end{pmatrix} e^{-0.0057656t} + \begin{pmatrix} -9.85754 \\ -8.70206 \end{pmatrix} e^{-0.0017344t} + \begin{pmatrix} 10.8 \\ 12.9 \end{pmatrix}.$$

This solution clearly shows that the trajectory converges asymptotically to the equilibrium solution as expected.

5. The predator-prey model is given by:

$$\begin{aligned}\frac{dH}{dt} &= 0.1H - 0.0005H^2 - 0.016HP &= f_1(H,P),\\ \frac{dP}{dt} &= 0.005HP - 0.2P &= f_2(H,P).\end{aligned}$$

The equilibria satisfy:

$$\begin{aligned} H_e(0.1 - 0.0005H_e - 0.016P_e) &= 0, \\ P_e(0.005H_e - 0.2) &= 0. \end{aligned}$$

One equilibrium is $(H_e, P_e) = (0, 0)$, the extinction equilibrium. When $P_e = 0$, then there is another equilibrium at $H_e = 200$, the carrying capacity. Finally, there is a third equilibrium with $0.005H_e - 0.2 = 0$ and $0.1 - 0.0005H_e - 0.016P_e = 0$. The first equation gives $H_e = 40$, which gives $P_e = 5$ in the second equation. Thus, there is a coexistence equilibrium, $(H_e, P_e) = (40, 5)$.

As we did in class, we use Taylor's theorem to linearize this system about the equilibria (finding the **Jacobian matrix**). If $h(t) = H(t) - H_e$ and $p(t) = P(t) - P_e$, then the linearized system can be written:

$$\begin{pmatrix} \dot{h} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1(H_e,P_e)}{\partial h} & \frac{\partial f_1(H_e,P_e)}{\partial p} \\ \frac{\partial f_2(H_e,P_e)}{\partial h} & \frac{\partial f_2(H_e,P_e)}{\partial p} \end{pmatrix} \begin{pmatrix} h \\ p \end{pmatrix} = \begin{pmatrix} 0.1 - 0.001H_e - 0.016P_e & -0.016H_e \\ 0.005P_e & 0.005H_e - 0.2 \end{pmatrix} \begin{pmatrix} h \\ p \end{pmatrix}.$$

The linear system about $(H_e, P_e) = (0, 0)$ is

$$\begin{pmatrix} \dot{h} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0.1 & 0 \\ 0 & -0.2 \end{pmatrix} \begin{pmatrix} h \\ p \end{pmatrix},$$

which has eigenvalues $\lambda_1 = -0.2$ with associated eigenvector $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\lambda_2 = 0.1$ with associated eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This is a **saddle node** at the extinction equilibrium. The general linear solution is given by

$$\begin{pmatrix} h(t) \\ p(t) \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-0.2t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{0.1t}.$$

Thus, if there are both predators and prey, then the solution moves away from extinction (unstable).

The linear system about $(H_e, P_e) = (200, 0)$ (carrying capacity of prey) is

$$\begin{pmatrix} \dot{h} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} -0.1 & -3.2 \\ 0 & 0.8 \end{pmatrix} \begin{pmatrix} h \\ p \end{pmatrix},$$

which has eigenvalues $\lambda_1 = -0.1$ with associated eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda_2 = 0.8$ with associated eigenvector $\mathbf{v}_2 = \begin{pmatrix} 32 \\ -9 \end{pmatrix}$. This is a **saddle node** at this equilibrium. The general linear solution is given by

$$\begin{pmatrix} h(t) \\ p(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-0.1t} + c_2 \begin{pmatrix} 32 \\ -9 \end{pmatrix} e^{0.8t}.$$

Thus, if there are both predators and prey near the carrying capacity, then the solution moves away from this equilibrium (unstable).

The linear system about $(H_e, P_e) = (40, 5)$ (coexistence) is

$$\begin{pmatrix} \dot{h} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} -0.02 & -0.64 \\ 0.025 & 0 \end{pmatrix} \begin{pmatrix} h \\ p \end{pmatrix}.$$

The characteristic equation is

$$\det\begin{vmatrix} -0.02-\lambda & -0.64 \\ 0.025 & -\lambda \end{vmatrix} = \lambda^2 + 0.02\lambda + 0.016 = 0,$$

which has eigenvalues $\lambda_{1,2} = -0.01 \pm 0.1261i$. Let $\omega = 0.1261$ then for $\lambda_1 = -0.01 + i\omega$, the associated eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 0.64 \\ -0.01 - i\omega \end{pmatrix}$, which leads to the general solution

$$\begin{pmatrix} h(t) \\ p(t) \end{pmatrix} = c_1 e^{-0.01t}\begin{pmatrix} 0.64\cos(\omega t) \\ -0.01\cos(\omega t) + \omega\sin(\omega t) \end{pmatrix} + c_2 e^{-0.01t}\begin{pmatrix} 0.64\sin(\omega t) \\ -0.01\sin(\omega t) - \omega\cos(\omega t) \end{pmatrix}.$$

This is a stable spiral, so all solutions of this model spiral into the coexistence equilibrium, $(H_e, P_e) = (40, 5)$.

6. The competition model is given by the system of differential equations:

$$\begin{aligned} \frac{dx_1}{dt} &= 0.3x_1 - 0.005x_1^2 - 0.009x_1x_2 &= g_1(x_1, x_2), \\ \frac{dx_2}{dt} &= 0.1x_2 - 0.0025x_2^2 - 0.002x_1x_2 &= g_2(x_1, x_2). \end{aligned}$$

The equilibria are found by solving:

$$\begin{aligned} g_1(x_{1e}, x_{2e}) &= x_{1e}(0.3 - 0.005x_{1e} - 0.009x_{2e}) &= 0, \\ g_2(x_{1e}, x_{2e}) &= x_{2e}(0.1 - 0.0025x_{2e} - 0.002x_{1e}) &= 0. \end{aligned}$$

This system has **4** equilibria. The trivial or **extinction** equilibrium is obvious, $(x_{1e}, x_{2e}) = (0, 0)$. When one of the populations is zero, then the other can go to its **carrying capacity**. Thus, when $x_{2e} = 0$, the equation $0.3 - 0.005x_{1e} = 0$ gives the carrying capacity of x_1 with the equilibrium $(x_{1e}, x_{2e}) = (60, 0)$. Similarly, when $x_{1e} = 0$, the equation $0.1 - 0.0025x_{2e} = 0$ gives the carrying capacity of x_2 with the equilibrium $(x_{1e}, x_{2e}) = (0, 40)$. The coexistence equilibrium satisfies

$$0.005x_{1e} + 0.009x_{2e} = 0.3 \qquad \text{and} \qquad 0.0025x_{2e} + 0.002x_{1e} = 0.1,$$

which gives $(x_{1e}, x_{2e}) \approx (27.273, 18.182)$.

As we did in the previous problem, we linearize and find the **Jacobian matrix** (using Taylor's theorem). If $y_1(t) = x_1(t) - x_{1e}$ and $y_2(t) = x_2(t) - x_{2e}$, then the linearized system can be written:

$$\begin{aligned} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} &= \begin{pmatrix} \frac{\partial g_1(x_{1e},x_{2e})}{\partial y_1 h} & \frac{\partial g_1(x_{1e},x_{2e})}{\partial y_2} \\ \frac{\partial g_2(x_{1e},x_{2e})}{\partial y_1} & \frac{\partial g_2(x_{1e},x_{2e})}{\partial y_2} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} &= \begin{pmatrix} 0.3 - 0.01x_{1e} - 0.009x_{2e} & -0.009x_{1e} \\ -0.002x_{2e} & 0.1 - 0.005x_{2e} - 0.002x_{1e} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \end{aligned}$$

The linear system about $(x_{1e}, x_{2e}) = (0, 0)$ is

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

which has eigenvalues $\lambda_1 = 0.1$ with associated eigenvector $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\lambda_2 = 0.3$ with associated eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This is an **unstable node** at the extinction equilibrium. The general linear solution is given by

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{0.1t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{0.3t}.$$

Thus, if there are any individuals of either species, then the solution moves away from extinction (unstable).

The linear system about $(x_{1e}, x_{2e}) = (60, 0)$ is

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -0.3 & -0.54 \\ 0 & -0.02 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

which has eigenvalues $\lambda_1 = -0.3$ with associated eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda_2 = -0.02$ with associated eigenvector $\mathbf{v}_2 = \begin{pmatrix} 27 \\ -14 \end{pmatrix}$. This is a **stable node** at this carrying capacity equilibrium. The general linear solution is given by

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-0.3t} + c_2 \begin{pmatrix} 27 \\ -14 \end{pmatrix} e^{-0.02t}.$$

Thus, near this equilibrium all solutions are attracted, leading to extinction of species x_2.

The linear system about $(x_{1e}, x_{2e}) = (0, 40)$ is

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -0.06 & 0 \\ -0.08 & -0.1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

which has eigenvalues $\lambda_1 = -0.1$ with associated eigenvector $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\lambda_2 = -0.06$ with associated eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. This is a **stable node** at this carrying capacity equilibrium. The general linear solution is given by

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-0.1t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-0.06t}.$$

Thus, near this equilibrium all solutions are attracted, leading to extinction of species x_1.

The linear system about $(x_{1e}, x_{2e}) = (27.273, 18.182)$ is

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -0.13636 & -0.24545 \\ -0.03636 & -0.04545 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

which has eigenvalues $\lambda_1 = -0.19575$ with associated eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0.24195 \end{pmatrix}$ and $\lambda_2 = 0.013932$ with associated eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -0.61232 \end{pmatrix}$. This is a **saddle node** at this coexistence equilibrium. The general linear solution is given by

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0.24195 \end{pmatrix} e^{-0.19575t} + c_2 \begin{pmatrix} 1 \\ -0.61232 \end{pmatrix} e^{0.013932t}.$$

Thus, near this equilibrium the solutions split and go away from the coexistence equilibrium. Depending on initial conditions the solution will eventually go toward one of the carrying capacity equilibria, leaving one species at carrying capacity and the other species extinct. This is known as **competitive exclusion**.