Math 337 - Elementary Differential Equations

Lecture Notes – Introduction to Differential Equations

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Outline

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Basic Information: Text/Topics

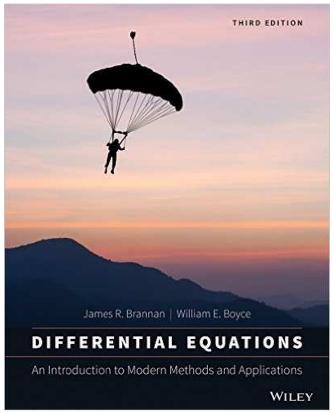
Text: The text is optional and old editions are fine.

Brannan and Boyce: Differential Equations: An Introduction to Modern Methods and Applications.

Wiley 2015.

ISBN 978-1-118-53177-8

Lecture Notes available at Bookstore



Basic Information: Text/Topics

1. Introductory Definitions
2. Qualitative Methods and Direction Fields
3. Linear Equations
4. Separable Equations
5. Exact and Bernoulli Equation
6. Existence and Uniquess
7. Numerical Methods
8. 2D Linear Systems
9. Second Order Differential Equations
10. Laplace Transforms
11. Power Series

SDSU

Other Differential Equation Courses

Differential Equations and Dynamical Systems: Several courses extend the material from this class. Courses from the Nonlinear Dynamical Systems Group.

- **Math 531:** ***Partial Differential Equations***
- **Math 537:** ***Ordinary Differential Equations***
- **Math 538:** ***Discrete Dynamical Systems and Chaos***
- **Math 542:** ***Introduction to Computational Ordinary Differential Equations***

SDSU

Basic Information: Grading

Approximate Grading

Homework, including WeBWorK	30%
Lecture Activities/Computer Labs	25%
3 Exams	27%
Final	18%

- Homework is done in WeBWorK and written problems (most inside WW problems) are submitted to Gradescope. Critical to **keep up** on HW after each lecture.
- Lecture Activities are written problems after lectures, which are submitted to Gradescope.
- Exams are based heavily on HW problems and examples from lectures.
- Final: Friday, May 6, 13:00 – 15:00

SDSU

Expectations and Procedures, I

- Most class attendance is OPTIONAL — Homework and announcements will be posted on the class web page. If/when you attend class:
  - You **must wear a mask** according to University rules.
  - Please be on time and pay attention.
  - Please turn off mobile phones.
  - Please be courteous to other students and the instructor.
  - Abide by university statutes, and all applicable local, state, and federal laws.

SDSU

Expectations and Procedures, II

- Please, turn in assignments on time. (The instructor reserves the right not to accept late assignments, and there is a maximum of **2** extensions of WeBWorK during the semester.)
- The instructor will make special arrangements for students with documented learning disabilities and will `try` to make accommodations for other unforeseen circumstances, *e.g.* illness, personal/family crises, etc. in a way that is fair to all students enrolled in the class. ***Please contact the instructor EARLY regarding special circumstances.***
- Students are expected ***and encouraged*** to ask questions in class!
- Students are expected ***and encouraged*** to to make use of office hours! If you cannot make it to the scheduled office hours: contact the instructor to schedule an appointment!

Expectations and Procedures, III

- Missed midterm exams: Don't miss exams! The instructor reserves the right to schedule make-up exams, modify the type and nature of this make-up, and/or base the grade solely on other work (including the final exam).
- Missed final exam: Don't miss the final! Contact the instructor ASAP or a grade of incomplete or F will be assigned.
- ***Academic honesty***: Submit your own work. Any cheating will be reported to University authorities and a **ZERO** will be given for that HW assignment or Exam.

MatLab

- Students can obtain **MatLab** from EDORAS Academic Computing.
- Google **SDSU MatLab** or access **https://edoras.sdsu.edu/ download/matlab.html**.
- **MatLab** and **Maple** can also be accessed in the **Computer Labs GMCS 421**, **422**, and **425**.
- A discounted student version of **Maple** is available.

Math 337: Formal Prerequisites

`Math 254` or `Math 342A` or `AE 280`

- These courses all require **Calculus 151**.
  - Assume good knowledge of ***differentiation*** and ***integration***.
  - Understand series techniques (especially ***Taylor's Theorem***)
  - Recall ***Partial Fractions Decomposition***.
- These courses all have sections on basic **Linear Algebra**.

Introduction

Introduction

- Differential equations frequently arise in modeling situations
- They describe population growth, chemical reactions, heat exchange, motion, and many other applications
- Differential equations are continuous analogs of discrete dynamical systems

Malthusian Growth 1

Discrete Malthusian Growth Model:

- Let the initial population, $P(t_0) = P_0$
- Define $t_n = t_0 + n\Delta t$ and $P_n = P(t_n)$
- Let r be the per capita growth rate per unit time
- The Discrete Malthusian Growth Model satisfies:

$$P_{n+1} = P_n + r\Delta t P_n = (1 + r\Delta t)P_n$$

- New population $=$ Old population $+$ per capita growth rate $\times$ length of time $\times$ Old population

Malthusian Growth 2

Discrete Malthusian Growth: $P_{n+1} = (1 + r\Delta t)P_n$, so

$$\begin{aligned}
P_1 &= (1 + r\Delta t)P_0 \\
P_2 &= (1 + r\Delta t)P_1 &= (1 + r\Delta t)^2 P_0 \\
P_3 &= (1 + r\Delta t)P_2 &= (1 + r\Delta t)^3 P_0 \\
&\vdots \\
P_n &= (1 + r\Delta t)P_{n-1} &= (1 + r\Delta t)^n P_0
\end{aligned}$$

The solution of this discrete model is

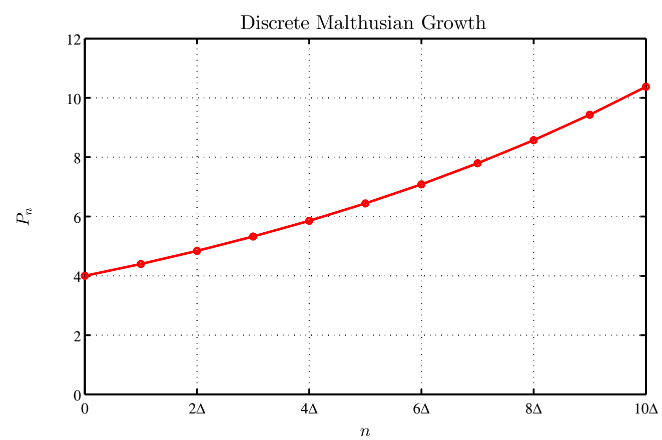
$$P_n = (1 + r\Delta t)^n P_0,$$

which is an exponential growth

Malthusian Growth 3

Discrete Malthusian Growth:

$$P_{n+1} = (1 + 0.1\Delta t)P_n \qquad P_0 = 4$$



Malthusian Growth 4

Malthusian Growth: Let $P(t)$ be the population at time $t = t_0 + n\Delta t$ and rearrange the model above

$$\begin{aligned} P_{n+1} - P_n &= r\Delta t P_n \\ P(t+\Delta t) - P(t) &= \Delta t \cdot rP(t) \\ \frac{P(t+\Delta t) - P(t)}{\Delta t} &= rP(t) \end{aligned}$$

Let Δt become very small

$$\lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = \frac{dP(t)}{dt} = rP(t),$$

which is a **Differential Equation**

Malthusian Growth 5

Solution of Malthusian Growth Model: The Malthusian growth model

$$\frac{dP(t)}{dt} = rP(t)$$

- The rate of change of a population is proportional to the population
- Let c be an arbitrary constant, so try a solution of the form

$$P(t) = ce^{kt}$$

- Differentiating

$$\frac{dP(t)}{dt} = cke^{kt},$$

which if $k = r$ is $rP(t)$, so satisfies the differential equation

Malthusian Growth 6

Solution of Malthusian Growth Model The Malthusian growth model satisfies

$$P(t) = ce^{rt}$$

- With the initial condition, $P(t_0) = P_0$, then the unique solution is

$$P(t) = P_0 e^{r(t-t_0)}$$

- Malthusian growth is often called exponential growth

Example 1: Malthusian Growth 1

Example 1: Malthusian Growth Consider the Malthusian growth model

$$\frac{dP(t)}{dt} = 0.02\,P(t) \quad \text{with} \quad P(0) = 100$$

Skip Example

- Find the solution
- Determine how long it takes for this population to double

Example 1: Malthusian Growth 2

Solution: The solution is given by

$$P(t) = 100\, e^{0.02t}$$

Since $P(0) = 100$, satisfying the initial condition, then by computing

$$\frac{dP}{dt} = 0.02(100\, e^{0.02t}) = 0.02\, P(t),$$

we find that this solution satisfies the differential equation

The population doubles when

$$\begin{aligned} 200 &= 100\, e^{0.02t} \\ 0.02t = \ln(2) \quad &\text{or} \quad t = 50\ln(2) \approx 34.66 \end{aligned}$$

Example 2: *E. coli* Study 1

Example 2: *E. coli* Study In this class we connect the ***ordinary differential equations (ODEs)*** to real world examples.

This requires fitting our model ODE to actual data.

Consider a culture of ***Escherichia coli*** growing in rich media at 25°C, which satisfies conditions for Malthusian growth,

$$\frac{dP}{dt} = kP, \qquad \text{with} \quad P(0) = P_0.$$

Below is a table of data:

t (min)	OD_{420}	t (min)	OD_{420}	t (min)	OD_{420}
0	0.195	60	0.308	120	0.473
20	0.206	80	0.364	140	0.527
40	0.241	100	0.421	160	0.618

Example 2: *E. coli* Study 2

Example 2: The most common means of fitting a model to data is ***minimizing the sum of square errors*** between the model and the data.

Basic statistics shows how to do fit data to a **straight line** (a common problem in multivariate Calculus).

In general this procedure is significantly more difficult and is usually done **numerically**.

Consider a set of $n+1$ **data points**: $(t_0, P_0), (t_1, P_1), ..., (t_n, P_n)$.

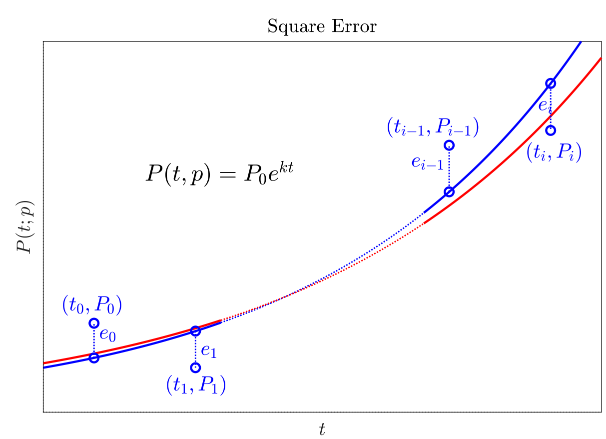
Assume the ***Malthusian growth model*** depends on some parameters, p:

$$P(t;p) = P_0 e^{kt}, \qquad \text{with} \quad p = [P_0, k],$$

which depends **nonlinearly** on the parameters, p.

Example 2: *E. coli* Study 3

Example 2: Below is a figure showing a data set and $P(t;p)$ with two different values of p, illustrating the computation of the ***sum of square errors***.



Example 2: *E. coli* Study 4

Least Squares Best Fit minimizes the square of the error in the distance between the P_i data values and the $P(t_i;p)$ value of the model.

The ***error*** between the data points and the model satisfies:

$$e_i = P_i - P(t_i;p) = P_i - P_0 e^{kt_i}, \qquad i = 0, ..., n,$$

which depends on P_0 and k.

The ***sums of square errors*** function depends on the parameters P_0 and k of the population model:

$$J(P_0,k) = \sum_{i=0}^{n} e_i^2 = \sum_{i=0}^{n} \left(P_i - P_0 e^{kt_i}\right)^2$$

The ***Least Squares Best Fit Model*** is the **minimum** of the function $J(P_0,k)$.

Example 2: *E. coli* Study 5

Least Squares Best Fit is found by setting the partials with respect to the parameters equal to zero:

$$\frac{\partial J(P_0,k)}{\partial P_0} = 0 \qquad \text{and} \qquad \frac{\partial J(P_0,k)}{\partial k} = 0.$$

These equations are highly nonlinear and difficult to solve in general.

Computer Software Packages often have numerical methods to approximate the solutions.

We examine two ***Computer Software Packages*** for finding the least squares best fit Model.

- **Excel's *Solver***
- **MatLab's *fminsearch***

Example 2: *E. coli* Study 6

Example 2: *E. coli* Study: Return to the **data** at the beginning of this study and the ***Malthusian growth model***, $P(t;p) = P_0 e^{kt}$.

There is a hyperlinked **Excel file** showing how this model is fit with ***Excel's Solver***

Below is the **MatLab code** for fitting the ***Malthusian ODE model***, finding the best initial condition, P_0, and growth rate, k.

The ***sum of square error program*** is given by:

```
function J = sumsq_ecoli(p,tdata,pdata)
% Function computing sum of square errors for ...
    Malthusian model
model = p(1)*exp(p(2)*tdata);
error = model - pdata;
J = error*error';
end
```


Example 2: *E. coli* Study 7

Example 2: *E. coli* Study: The ***sum of square error program*** is used inside the primary plotting program (line 11):

```
1  clear                    % Clear previous definitions
2  figure(1)                % Assign figure number
3  clf                      % Clear previous figures
4  hold off                 % Start with fresh graph
5  mytitle = '\it Escherichia coli';
6  xlab = '$t$ (min)';
7  ylab = '$P(t)$ (OD$_{420}$)';
8  td = [0 20  40  60  80  100 120 140 160];
9  pd = [0.195 0.206 0.241 0.308 0.364 0.421 0.473 ...
       0.527 0.618];
10 tt = linspace(0,180,200);
11 [p1,J,flag] = ...
       fminsearch(@sumsq_ecoli,[0.2,0.01],[],td,pd)
12 Pt = p1(1)*exp(p1(2)*tt);
```

Example 2: *E. coli* Study 8

Example 2: *E. coli* Study: The primary plotting program continues with:

```
plot(tt,Pt,'b-','LineWidth',1.5);
hold on
plot(td,pd,'bo','LineWidth',1.5);
grid
myeqn=['$P(t)=',num2str(p1(1)), 'e^{' ...
    ,num2str(p1(2)), 't}$'];
text(42,0.53,myeqn,'FontSize',14,'interpreter','latex');
legend('Model', 'Data','Location','southeast');
xlim([0 180]);
ylim([0 0.8]);
fontlabs = 'Times New Roman';
xlabel(xlab,'FontSize',14,'FontName',fontlabs, ...
    'interpreter','latex');
```


Example 2: *E. coli* Study 9

Example 2: *E. coli* Study: The primary plotting program finishes with:

```
ylabel(ylab,'FontSize',14,'FontName',fontlabs, ...
    'interpreter','latex');
title(mytitle,'FontSize',18,'FontName',fontlabs, ...
    'interpreter','latex');
set(gca,'FontSize',12);
print -depsc ecoli.eps
print -djpeg ecoli.jpg
```

Significantly, inside the above program we see the line:

```
[p1,J,flag] = ...
fminsearch(@sumsq_ecoli,[0.2,0.01],[],td,pd)
```

which invokes ***MatLab's nonlinear solver*** to **minimize** the ***sum of square errors***.

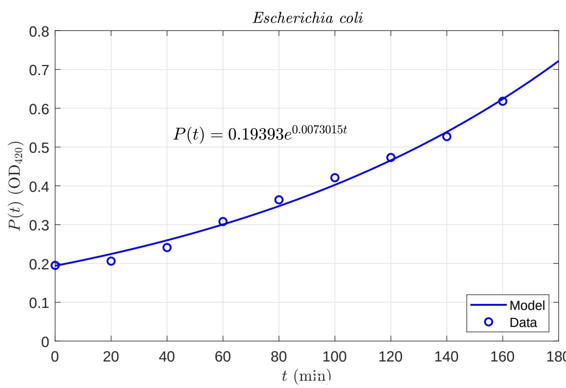
Example 2: *E. coli* Study 10

Example 2: *E. coli* Study: The result gives the ***best fitting model***:

$$P(t) = 0.19393e^{0.0073015t},$$

with a ***least sum of square errors***, $J = 0.0015809$.

The ***best fitting model*** with the **data** are shown in the graph below:



Example 2: *E. coli* Study 11

Example 2: *E. coli* Discrete Model: Fitting the solution to an **ODE** is a basic ***curve fitting*** exercise.

Finding the parameters for a ***discrete dynamical model*** is less straightforward.

The ***discrete Malthusian growth model*** is the rare discrete model with an explicit answer, so could be solved using a curve fitting routine.

Below we provide the **MatLab program** for fitting this discrete model through a ***least squares best fit*** using an iterated simulation of the model as parameters vary.

The ***discrete Mathusian growth model*** satisfies the equation:

$$P_{n+1} = (1+r)P_n, \qquad \text{with initial parameter} \quad P_0,$$

where n is the discrete time interval of 20 min.

Again we are minimizing the ***sum of square errors*** between the **model** and the **data** in the table above as the parameters, r and P_0, vary.

Example 2: *E. coli* Study 12

Example 2: *E. coli* Discrete Model: The model is simulated,

$$P_{n+1} = (1+r)P_n, \qquad \text{with initial parameter} \quad P_0,$$

for some parameters r and P_0, and its square error is computed with the following MatLab program:

```
function J = ec_disc_lst(p0,tdata,pdata)
% Least Squares fit to Malthusian Growth
N = length(tdata);
p = p0(1);
pop = [p];
err = [pdata(1) - p];
for i = 2:N  % Malthusian iteration
    p = p*(1+p0(2));
    pop = [pop,p];
    err = [err, pdata(i) - p];
end
J = err*err'; % Sum of square errors
end
```


Example 2: *E. coli* Study 13

Example 2: *E. coli* Discrete Model: The **main program** used to compute the best fitting parameters and create a graph is similar to the previous plotting program but varies in a few lines:

```
td = [0 20  40  60  80  100 120 140 160];
pd = [0.195 0.206 0.241 0.308 0.364 0.421 0.473 ...
    0.527 0.618];
[p1,J,flag] = ...
    fminsearch(@ec_disc_lst,[0.2,0.15],[],td,pd)
N = length(td);
p = p1(1);
pop = [p];
for i = 2:N+1  % Malthusian iteration
    p = p*(1+p1(2));
    pop = [pop,p];
end
```


Example 2: *E. coli* Study 14

Example 2: *E. coli* Discrete Model: The primary plotting program continues with:

```
plot([td,180],pop,'ro-','LineWidth',1.5);
hold on
plot(td,pd,'bo','LineWidth',1.5);
grid
myeqn=['$P_{n+1} = (1+',num2str(p1(2)), ...
    ')P_n,\quad P_0 =$', num2str(p1(1))];
text(37,0.63,myeqn,'FontSize',14,'interpreter','latex');
legend('Model', 'Data','Location','southeast');
```

As before, inside the above program the line:

```
[p1,J,flag] = ...
fminsearch(@ec_disc_lst,[0.2,0.15],[],td,pd)
```

invokes MatLab's nonlinear solver, which minimizes the sum of square errors.

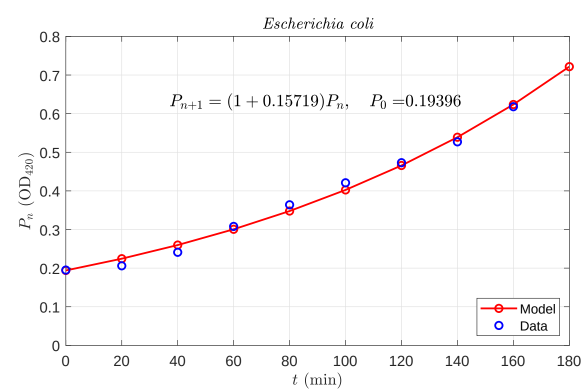
Example 2: *E. coli* Study 15

Example 2: *E. coli* Discrete Model: The result gives the ***best fitting model***:

$$P_{n+1} = 1.15719 P_n, \qquad P_0 = 0.19396,$$

with a ***least sum of square errors***, $J = 0.0015809$.

The ***best fitting model*** with the **data** are shown in the graph below:



What is a Differential Equation?

What is a Differential Equation?

Definition (Differential Equation)

An equation that contains derivatives of one or more unknown functions with respect to one or more independent variables is said to be a **differential equation**.

- The classical example is Newton's Law of motion
  - The mass of an object times its acceleration is equal to the sum of the forces acting on that object
  - Acceleration is the first derivative of velocity or the second derivative of position
- In biology, a differential equation describes a growth rate, a reaction rate, or the change in some physiological state

Types of Differential Equations

- This course considers **Ordinary Differential Equations**, where the **unknown function and its derivatives** depend on a single **independent variable**
- Mathematical physics often needs **Partial Differential Equations**, where the **unknown function and its derivatives** depend on two or more **independent variables**
  - **Example: Heat Equation**

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2}$$

- This course also examines some **Systems of Ordinary Differential Equations**, where there are several interacting **unknown functions and their derivatives** each depending on a single **independent variable**

Classification

Definition (Order)

The order of a **differential equation** matches the order of the highest derivative that appears in the equation.

Definition (Linear Differential Equation)

An n^{th} order ordinary differential equation $F(t, y, y', ..., y^{(n)}) = 0$ is said to be linear if it can be written in the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + ... + a_n(t)y = g(t).$$

The functions a_0, a_1, ... a_n, called the **coefficients** of the equation, can depend at most on the independent variable t. This equation is said to be homogeneous if the function $g(t)$ is zero for all t. Otherwise, the equation is nonhomogeneous.

Applications of Differential Equations 1

Radioactive Decay: Let $R(t)$ be the amount of a radioactive substance

- Radioactive elements transition through decay into another state at a rate proportional to the amount of radioactive material present
- The differential equation is

$$\frac{dR(t)}{dt} = -k\,R(t) \quad \text{with} \quad R(0) = R_0$$

- This is a first order, linear, homogeneous differential equation
- Like the Malthusian growth model, this has an exponential solution

$$R(t) = R_0 e^{-kt}$$

Applications of Differential Equations 2

Harmonic Oscillator: A Hooke's law spring exerts a force that is proportional to the displacement of the spring

- Newton's law of motion: Mass times the acceleration equals the force acting on the mass
- The simplest spring-mass problem is

$$my'' = -cy \qquad \text{or} \qquad y'' + k^2y = 0$$

- This is a **second order**, **linear**, **homogeneous differential equation**
- The general solution is

$$y(t) = c_1\cos(kt) + c_2\sin(kt),$$

where c_1 and c_2 are arbitrary constants

Applications of Differential Equations 3

Swinging Pendulum: A pendulum is a mass attached at one point so that it swings freely under the influence of gravity

- Newton's law of motion (ignoring resistance) gives the differential equation

$$my'' + g\sin(y) = 0$$

- The variable y is the angle of the pendulum, m is the mass of the bob of the pendulum, and g is the gravitational constant
- This is a **second order**, **nonlinear**, **homogeneous differential equation**
- This problem does not have an easily expressible solution

Applications of Differential Equations 4

Logistic Growth: Most populations are limited by food, space, or waste build-up, thus, cannot continue to grow according to Malthusian growth

- The Logistic growth model has a Malthusian growth term and a term limiting growth due to crowding
- The differential equation is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right)$$

- P is the population, r is the Malthusian rate of growth, and M is the carrying capacity of the population
- This is a **first order**, **nonlinear**, **homogeneous differential equation**
- We solve this problem later in the semester

Applications of Differential Equations 5

The van der Pol Oscillator: In electrical circuits, diodes show a rapid rise in current, leveling of the current, then a steep decline

- Biological applications include a similar approximation for nerve impulses
- The van der Pol Oscillator satisfies the differential equation

$$v'' + a(v^2 - 1)v' + v = b$$

- v is the voltage of the system, and a and b are constants
- This is a **second order**, **nonlinear**, **nonhomogeneous differential equation**
- This problem does not have an easily expressible solution, but shows interesting oscillations

Applications of Differential Equations 6

Lotka-Volterra – Predator and Prey Model: Model for studying the dynamics of predator and prey interacting populations

- Model for the population dynamics when one predator species and one prey species are tightly interconnected in an ecosystem
- System of differential equations

$$\begin{aligned} x' &= a\,x - b\,xy \\ y' &= -c\,y + d\,xy \end{aligned}$$

- x is the prey species, and y is the predator species
- This is a **system of first order**, **nonlinear**, **homogeneous differential equations**
- No explicit solution, but we'll study its behavior

Applications of Differential Equations 7

Forced Spring-Mass Problem with Damping: An extension of the spring-mass problem that includes viscous-damping caused by resistance to the motion and an external forcing function that is applied to the mass

- The model is given by

$$my'' + cy' + ky = F(t)$$

- y is the position of the mass, m is the mass of the object, c is the damping coefficient, k is the spring constant, $F(t)$ is an externally applied force
- This is a **second order**, **linear**, **nonhomogeneous differential equation**
- We'll learn techniques for solving this

Damped Spring-Mass Problem 1

Damped Spring-Mass Problem: Assume a mass attached to a spring with resistance satisfies the second order linear differential equation

$$y''(t) + 2y'(t) + 5y(t) = 0$$

Skip Example

Show that one solution to this differential equation is

$$y_1(t) = 2\,e^{-t}\sin(2t)$$

Damped Spring-Mass Problem 2

Solution: Damped spring-mass problem

- The 1^{st} derivative of $y_1(t) = 2\,e^{-t}\sin(2t)$

$$y_1'(t) = 2e^{-t}(2\cos(2t)) - 2e^{-t}\sin(2t) = 2e^{-t}(2\cos(2t) - \sin(2t))$$

- The 2^{nd} derivative of $y_1(t) = 2\,e^{-t}\sin(2t)$

$$\begin{aligned} y_1''(t) &= 2e^{-t}(-4\sin(2t) - 2\cos(2t)) - 2e^{-t}(2\cos(2t) - \sin(2t)) \\ &= -2e^{-t}(4\cos(2t) + 3\sin(2t)) \end{aligned}$$

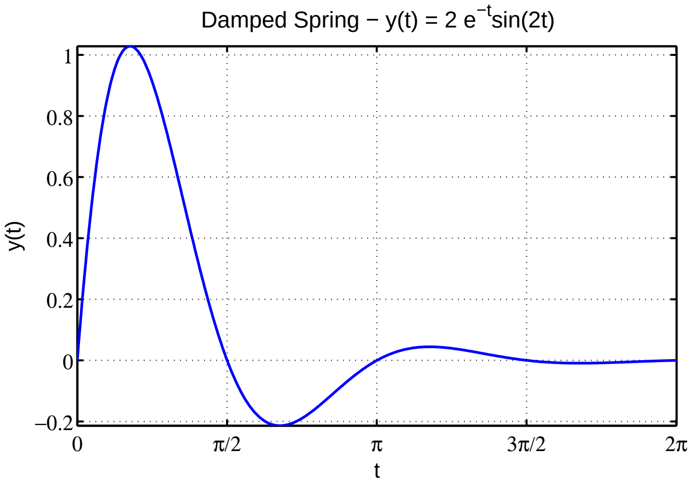
- Substitute into the spring-mass problem

$$\begin{aligned} y_1'' + 2y_1' + 5y &= -2e^{-t}(4\cos(2t) + 3\sin(2t)) \\ &\quad +2(2e^{-t}(2\cos(2t) - \sin(2t))) + 5(2e^{-t}\sin(2t)) \\ &= 0 \end{aligned}$$

It is often **easy** to check that a solution satisfies a differential equation.

Damped Spring-Mass Problem 3

Graph of Damped Oscillator



Initial Value Problem

Definition (Initial Value Problem)

An initial value problem for an n^{th} order differential equation

$$y^{(n)} = f(t, y, y', y'', ..., y^{(n-1)})$$

on an interval I consists of this differential equation together with n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad ..., \quad y^{(n-1)}(t_0) = y_{n-1}$$

prescribed at a point $t_0 \in I$, where y_0, y_1, ..., y_{n-1} are given constants.

Under reasonable conditions the solution of an **Initial Value Problem** has a unique solution.

Evaporation Example 1

Evaporation Example: Animals lose moisture proportional to their surface area

Skip Example

- If $V(t)$ is the volume of water in the animal, then the moisture loss satisfies the differential equation

$$\frac{dV}{dt} = -0.03\, V^{2/3}, \qquad V(0) = 8 \text{ cm}^3$$

- The initial amount of water is 8 cm^3 with t in days
- Verify the solution is

$$V(t) = (2 - 0.01t)^3$$

- Determine when the animal becomes totally desiccated according to this model
- Graph the solution

Evaporation Example 2

Solution: Show $V(t) = (2 - 0.01t)^3$ satisfies

$$\frac{dV}{dt} = -0.03\, V^{2/3}, \qquad V(0) = 8 \text{ cm}^3$$

- $V(0) = (2 - 0.01(0))^3 = 8$, so satisfies the initial condition
- Differentiate $V(t)$,

$$\frac{dV}{dt} = 3(2 - 0.01t)^2(-0.01) = -0.03(2 - 0.01t)^2$$

- But $V^{2/3}(t) = (2 - 0.01t)^2$, so

$$\frac{dV}{dt} = -0.03\, V^{2/3}$$

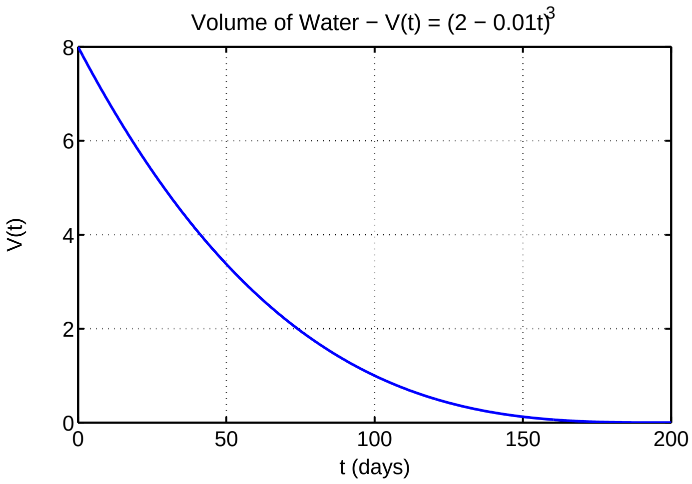
Evaporation Example 3

Solution (cont): Find the time of total desiccation

- Must solve
$$V(t) = (2 - 0.01t)^3 = 0$$
- Thus,
$$2 - 0.01t = 0 \qquad \text{or} \qquad t = 200$$
- It takes 200 days for complete desiccation

Evaporation Example 4

Graph of Desiccation



Nonautonomous Example 1

Nonautonomous Example: Consider the nonautonomous differential equation with initial condition (**Initial Value Problem**):
$$\frac{dy}{dt} = -ty^2, \qquad y(0) = 2$$

- Show that the solution to this differential equation, including the initial condition, is
$$y(t) = \frac{2}{t^2+1}$$
- Graph of the solution

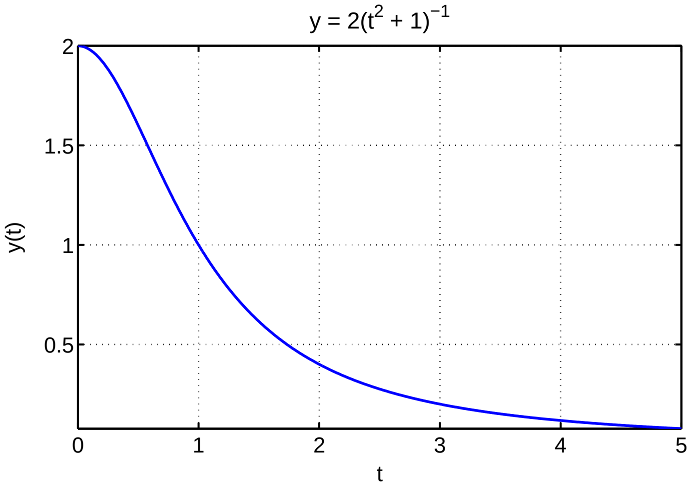
Nonautonomous Example 2

Solution: Consider the solution
$$y(t) = \frac{2}{t^2+1} = 2(t^2+1)^{-1}$$

- The initial condition is
$$y(0) = \frac{2}{0^2+1} = 2$$
- Differentiate $y(t)$,
$$\frac{dy}{dt} = -2(t^2+1)^{-2}(2t) = -4t(t^2+1)^{-2}$$
- However,
$$-ty^2 = -t(2(t^2+1)^{-1})^2 = -4t(t^2+1)^{-2}$$
- Thus, the differential equation is satisfied

Nonautonomous Example 3

Solution of Nonautonomous Differential Equation



Introduction to Maple 1

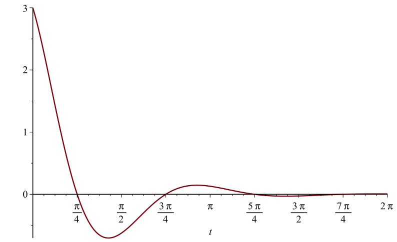
Introduction to Maple: A Symbolic Math Program

We enter a function $y(t) = 3e^{-t}\cos(2t)$,

$y := t \to 3\cdot\exp(-t)\cdot\cos(2\cdot t);$

The arrow is $-$ and $>$ and multiplication is $*$. To plot this function

$plot(y(t), t = 0..2\cdot\text{Pi});$



Introduction to Maple 2

We have the function: $y(t) = 3e^{-t}\cos(2t)$,

This can be differentiated (and stored in variable dy) by typing

$dy := diff(y(t), t);$

Maple gives:

$$dy := -3e^{-t}\cos(2t) - 6e^{-t}\sin(2t)$$

The absolute minimum and a relative maximum are found with Maple:

$tmin := fsolve(dy = 0, t = 1..2); \quad y(tmin);$
$tmax := fsolve(dy = 0, t = 2.5..3.5); \quad y(tmax);$

The result was an **absolute minimum** at $(1.33897, -0.703328)$.
The result was a **relative maximum** at $(2.90977, 0.1462075)$.

Introduction to Maple 3

With $y(t) = 3e^{-t}\cos(2t)$, we can solve

$$\int 3e^{-t}\cos(2t)dt \qquad \text{and} \qquad \int_0^5 3e^{-t}\cos(2t)dt$$

These can be integrated by typing

$int(y(t), t); \quad int(y(t), t = 0..5); \quad evalf(\%);$

For the indefinite integral, Maple gives:

$$-\frac{3}{5}e^{-t}\cos(2t) + \frac{6}{5}e^{-t}\sin(2t)$$

For the definite integral, Maple gives:

$$\frac{3}{5} - \frac{3}{5}e^{-5}\cos(10) + \frac{6}{5}e^{-5}\sin(10) \quad = \quad 0.59899347$$

Introduction to Maple

Show $y(t) = 3e^{-t}\cos(2t)$ is a solution of the differential equation

$$y'' + 2y' + 5y = 0.$$

The function and derivatives are entered by

$y := t \to 3 \cdot \exp(-t) \cdot \cos(2 \cdot t);$
$dy := \mathit{diff}(y(t), t);$
$sdy := \mathit{diff}(y(t), t\$2);$

If we type

$sdy + 2 \cdot dy + 5 \cdot y(t);$

Maple gives **0**, which verifies this is a solution.

Introduction to Maple

Maple finds the general solution to the differential equation

$de := \mathit{diff}(Y(t), t\$2) + 2 \cdot \mathit{diff}(Y(t), t) + 5 \cdot Y(t) = 0;$
$dsolve(de, Y(t));$

Maple produces

$$Y(t) = C_1 e^{-t}\sin(2t) + C_2 e^{-t}\cos(2t)$$

To solve an initial value problem, say $Y(0) = 2$ and $Y'(0) = -1$, enter

$dsolve(\{de, Y(0) = 2, D(Y)(0) = -1\}, Y(t));$

Maple produces

$$Y(t) = \frac{1}{2}e^{-t}\sin(2t) + 2e^{-t}\cos(2t),$$

which is made into a useable function by typing

$Y := unapply(rhs(\%), t);$