Math 337 - Elementary Differential Equations

Lecture Notes – Exact and Bernoulli Differential Equations

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Outline

Introduction

Introduction

- Exact Differential Equations
  - Potential Functions
  - Gravity
- Bernoulli's Differential Equation
- Applications
  - Logistic Growth

Exact Differential Equations

Exact Differential Equations - Potential functions

- In physics, **conservative forces** lead to **potential functions**, where no work is performed on a closed path
- Alternately, the work is independent of the path
- Potential functions arise as solutions of Laplace's equation in PDEs
- Potential function are **analytic functions** in Complex Variables
- Naturally arise from implicit differentiation

Gravity

Gravity

- The force of gravity between two objects mass m_1 and m_2 satisfy

$$F(x,y) = Gm_1m_2\left(\frac{x\mathbf{i}}{(x^2+y^2)^{3/2}} + \frac{y\mathbf{j}}{(x^2+y^2)^{3/2}}\right)$$

- The potential energy satisfies

$$U(x,y) = -\frac{Gm_1m_2}{(x^2+y^2)^{1/2}}$$

- Perform **Implicit differentiation** on $U(x,y)$, where we let y depend on x (path $y(x)$ depends on x):

$$\frac{dU(x,y)}{dx} = Gm_1m_2\left(\frac{x}{(x^2+y^2)^{3/2}} + \left(\frac{y}{(x^2+y^2)^{3/2}}\right)\frac{dy}{dx}\right)$$

- A conservative function satisfies $\dfrac{dU}{dx} = 0$

Gravity

Differential Equation for Gravity

- The differential equation for gravity is

$$Gm_1m_2\left(\frac{x}{(x^2+y^2)^{3/2}} + \left(\frac{y}{(x^2+y^2)^{3/2}}\right)\frac{dy}{dx}\right) = 0$$

- By the way this problem was set up, the **solution** is the implicit **potential function**

$$U(x,y(x)) = -\frac{Gm_1m_2}{(x^2+y^2(x))^{1/2}} = C$$

Gravity

Potential Function

- Consider a *potential function*, $\phi(x,y)$
- By *implicit differentiation*

$$\frac{d\phi(x,y)}{dx} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx}$$

- If the *potential function* satisfies $\phi(x,y) = C$ (*level potential field*), then

$$\frac{d\phi(x,y)}{dx} = 0$$

- This gives rise to an **Exact differential equation**

Gravity

Definition

Suppose there is a function $\phi(x,y)$ with

$$\frac{\partial\phi}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = N(x,y).$$

The first-order differential equation given by

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is an **exact differential equation** with the *implicit solution* satisfying:

$$\phi(x,y) = C.$$

Example 1

Example: Consider the differential equation:

$$(2x + y\cos(xy)) + (4y + x\cos(xy))\frac{dy}{dx} = 0$$

This equation is clearly nonlinear and not separable.
We **hope** that it might be **exact**!

If it is **exact**, then there must be a *potential function*, $\phi(x,y)$ satisfying:

$$\frac{\partial\phi}{\partial x} = 2x + y\cos(xy) \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = 4y + x\cos(xy).$$

Example 2

Example (cont): Begin with

$$\frac{\partial\phi}{\partial x} = M(x,y) = 2x + y\cos(xy).$$

Integrate this with respect to x, so

$$\phi(x,y) = \int (2x + y\cos(xy))dx = x^2 + \sin(xy) + h(y),$$

where $h(y)$ is some function depending only on y

Similarly, we want

$$\frac{\partial\phi}{\partial y} = N(x,y) = 4y + x\cos(xy).$$

Integrate this with respect to y, so

$$\phi(x,y) = \int (4y + x\cos(xy))dy = 2y^2 + \sin(xy) + k(x),$$

where $k(x)$ is some function depending only on x

Example 3

Example (cont): The *potential function*, $\phi(x,y)$ satisfies

$$\phi(x,y) = x^2 + \sin(xy) + h(y) \qquad \text{and} \qquad \phi(x,y) = 2y^2 + \sin(xy) + k(x)$$

for some $h(y)$ and $k(x)$

Combining these results yields the solution

$$\phi(x,y) = x^2 + 2y^2 + \sin(xy) = C.$$

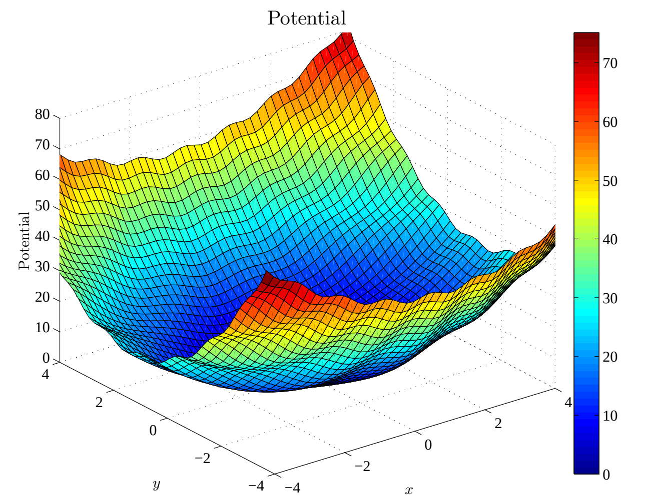
Implicit differentiation yields:

$$\frac{d\phi}{dx} = (2x + y\cos(xy)) + (4y + x\cos(xy))\frac{dy}{dx} = 0,$$

the original differential equation.

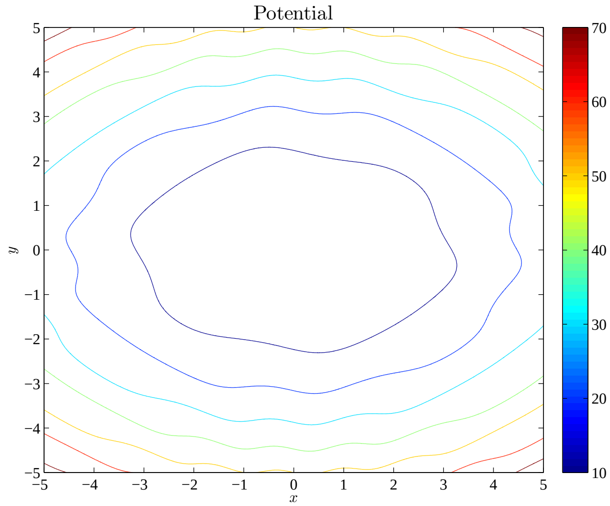
Potential Example

Graph of the Potential Function



Potential Example

Contour of the Potential Function



Exact Differential Equation

Theorem

Let the functions M, N, M_y, and N_x (subscripts denote partial derivatives) be continuous in a rectangular region $R : \alpha < x < \beta, \gamma < y < \delta$. Then the DE

$$M(x, y) + N(x, y)y' = 0$$

is an exact differential equation in R if and only if

$$M_y(x, y) = N_x(x, y)$$

at each point in R. Furthermore, there exists a potential function $\phi(x, y)$ solving this differential equation with

$$\phi_x(x, y) = M(x, y) \qquad \phi_y(x, y) = N(x, y).$$

Example

Consider the differential equation

$$2t\cos(y) + 2 + (2y - t^2\sin(y))y' = 0$$

Since

$$\frac{\partial M(t, y)}{\partial y} = -2t\sin(y) = \frac{\partial N(t, y)}{\partial t},$$

this DE is **exact**

Integrating

$$\begin{aligned} \int (2t\cos(y) + 2)dt &= t^2\cos(y) + 2t + h(y) \quad \text{and} \\ \int (2y - t^2\sin(y))dy &= y^2 + t^2\cos(y) + k(t) \end{aligned}$$

It follows that the potential function is

$$\phi(t, y) = y^2 + 2t + t^2\cos(y) = C$$

Logistic Growth Equation

Logistic Growth Equation is one of the most important population models

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right), \qquad P(0) = P_0$$

This a 1^{st} order nonlinear differential equation

It is separable, so can be written:

$$\int \frac{dP}{P\left(\frac{P}{M} - 1\right)} = -\int r dt = -rt + C$$

Left integral requires partial fractions composition

$$\frac{1}{P\left(\frac{P}{M} - 1\right)} = \frac{A}{P} + \frac{B}{\left(\frac{P}{M} - 1\right)}$$

Logistic Growth Equation 2

Fundamental Theorem of Algebra gives $A = -1$ and $B = 1/M$, so integrals become

$$\int \frac{(1/M)}{\left(\frac{P}{M} - 1\right)} dP - \int \frac{dP}{P} = -rt + C$$

With a substitution, we have

$$\ln\left(\frac{P(t)}{M} - 1\right) - \ln(P(t)) = \ln\left(\frac{P(t) - M}{MP(t)}\right) = -rt + C$$

Exponentiating (with $K = e^C$)

$$\frac{P(t) - M}{MP(t)} = Ke^{-rt} \qquad \text{or} \qquad P(t) = \frac{M}{1 - KMe^{-rt}}$$

Logistic Growth Equation 3

Logistic Growth Equation with initial condition is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right), \qquad P(0) = P_0$$

With the initial condition and some algebra, the **solution** is

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-rt}}$$

This solution took lots of work!

Bernoulli - Logistic Growth Equation 1

Alternate Solution - Logistic Growth Equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right), \qquad P(0) = P_0$$

This is rewritten

$$\frac{dP}{dt} - rP = -\frac{r}{M}P^2$$

Consider a substitution $u = P^{1-2} = P^{-1}$, so $\frac{du}{dt} = -P^{-2}\frac{dP}{dt}$

Multiply the logistic equation by $-P^{-2}$, so

$$-P^{-2}\frac{dP}{dt} + rP^{-1} = \frac{r}{M}$$

or

$$\frac{du}{dt} + ru = \frac{r}{M}$$

Bernoulli - Logistic Growth Equation 2

Alternate Solution (cont): With the substitution $u(t) = \frac{1}{P(t)}$, the new DE is

$$\frac{du}{dt} + ru = \frac{r}{M},$$

which is a **Linear Differential Equation**

With our linear techniques, the integrating factor is $\mu(t) = e^{rt}$, so

$$\frac{d}{dt}\left(e^{rt}u(t)\right) = \frac{r}{M}e^{rt}$$

so

$$e^{rt}u(t) = \frac{e^{rt}}{M} + C \qquad \text{or} \qquad u(t) = \frac{1}{M} + Ce^{-rt}$$

or

$$\frac{1}{P(t)} = \frac{1}{M} + Ce^{-rt}$$

Bernoulli - Logistic Growth Equation 3

Alternate Solution (cont): Inverting this gives

$$P(t) = \frac{M}{1 + MCe^{-rt}}$$

The initial condition $P(0) = P_0$, so $P_0 = \frac{M}{1+MC}$ or

$$C = \frac{M - P_0}{P_0 M}$$

It follows that

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-rt}}$$

This solution is MUCH easier!

Bernoulli's Equation 1

Definition

A differential equation of the form

$$\frac{dy}{dt} + q(t)y = r(t)y^n,$$

where n is any real number, is called a **Bernoulli's equation**

Define $u = y^{1-n}$, so

$$\frac{du}{dt} = (1 - n)y^{-n}\frac{dy}{dt}$$

Bernoulli's Equation 2

The substitution $u = y^{1-n}$ suggests multiply by $(1 - n)y^{-n}$, changing **Bernoulli's Equation** to

$$(1 - n)y^{-n}\frac{dy}{dt} + (1 - n)q(t)y^{1-n} = (1 - n)r(t),$$

which results in the new equation

$$\frac{du}{dt} + (1 - n)q(t)u = (1 - n)r(t)$$

This is a 1^{st} **order linear differential equation**, which is easy to solve

Example: Bernoulli's Equation 1

Example: Consider the Bernoulli's equation:

$$3t\frac{dy}{dt} + 9y = 2ty^{5/3}$$

Solution: Rewrite the equation

$$\frac{dy}{dt} + \frac{3}{t}y = \frac{2}{3}y^{5/3}$$

and use the substitution $u = y^{1-5/3} = y^{-2/3}$ with $\frac{du}{dt} = -\frac{2}{3}y^{-5/3}\frac{dy}{dt}$

Multiply equation above by $-\frac{2}{3}y^{-5/3}$ and obtain

$$\frac{du}{dt} - \frac{2}{t}u = -\frac{4}{9},$$

which is a **linear differential equation**

Example: Bernoulli's Equation 2

Example (cont): The **linear differential equation** in $u(t)$ is

$$\frac{du}{dt} - \frac{2}{t}u = -\frac{4}{9},$$

which has an integrating factor

$$\mu(t) = e^{-2\int \frac{dt}{t}} = e^{-2\ln(t)} = \frac{1}{t^2}$$

This gives

$$\frac{d}{dt}\left(\frac{u}{t^2}\right) = -\frac{4}{9t^2},$$

which integrating gives

$$\frac{u}{t^2} = \frac{4}{9t} + C \qquad \text{or} \qquad u(t) = \frac{4t}{9} + Ct^2$$

Example: Bernoulli's Equation 3

Example (cont): However, $u(t) = y^{-2/3}(t)$, so if

$$u(t) = \frac{4t}{9} + Ct^2, \qquad \text{then} \qquad y^{-2/3}(t) = \frac{4t}{9} + Ct^2$$

The explicit solution is

$$y(t) = \left(\frac{9}{4t + 9Ct^2}\right)^{\frac{3}{2}}$$