Math 337 - Elementary Differential Equations

Lecture Notes – Direction Fields and Phase Portraits - 1D

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Outline

Solution of Linear Growth and Decay Models

Previously showed that for **Malthusian growth** or **Radioactive decay** the ***linear differential equation***:

$$\frac{dy}{dt} = a\,y \qquad \text{with} \qquad y(0) = y_0,$$

has the solution:

$$y(t) = y_0 e^{at}.$$

More generally, we have the following solution:

Method (General Solution to Linear Growth and Decay Models)

Consider

$$\frac{dy}{dt} = a\,y \qquad \text{with} \qquad y(t_0) = y_0.$$

The solution is

$$y(t) = y_0 e^{a(t-t_0)}.$$

Example: Linear Decay Model

Example: Linear Decay Model: Consider

$$\frac{dy}{dt} = -0.3\,y \qquad \text{with} \qquad y(4) = 12$$

The solution is

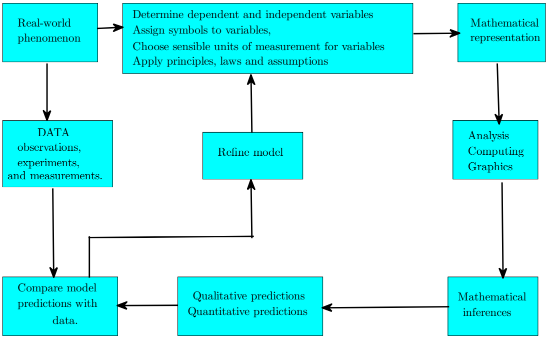
$$y(t) = 12\,e^{-0.3(t-4)}$$

This solution shows a substance decaying at a rate $k = 0.3$ starting with 12 units of substance y.

However, the solution is ***shifted (horizontally)*** by 4 units of time.

Mathematical Modeling

A diagram of the Modeling Process



Newton's Law of Cooling 1

Newton's Law of Cooling:

- After a murder (or death by other causes), the forensic scientist takes the temperature of the body
- Later the temperature of the body is taken again to find the rate at which the body is cooling
- Two (or more) data points are used to extrapolate back to when the murder occurred
- This property is known as **Newton's Law of Cooling**

Newton's Law of Cooling 2

Newton's Law of Cooling states that the rate of change in temperature of a cooling body is proportional to the difference between the temperature of the body and the surrounding environmental temperature

- If $T(t)$ is the temperature of the body, then it satisfies the differential equation

$$\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T(0) = T_0$$

- The parameter k is dependent on the specific properties of the particular object (body in this case)
- T_e is the environmental temperature
- T_0 is the initial temperature of the object

Murder Example 1

Murder Example

- Suppose that a murder victim is found at 8:30 am
- The temperature of the body at that time is 30°C
- Assume that the room in which the murder victim lay was a constant 22°C
- Suppose that an hour later the temperature of the body is 28°C
- Normal temperature of a human body when it is alive is 37°C
- Use this information to determine the approximate time that the murder occurred

Murder Example 2

Solution: From the model for Newton's Law of Cooling and the information that is given, if we set $t = 0$ to be 8:30 am, then we solve the initial value problem

$$\frac{dT}{dt} = -k(T(t) - 22) \quad \text{with} \quad T(0) = 30$$

- Make a change of variables $z(t) = T(t) - 22$
- Then $z'(t) = T'(t)$, so the differential equation above becomes

$$\frac{dz}{dt} = -kz(t), \quad \text{with} \quad z(0) = T(0) - 22 = 8$$

- This is the radioactive decay problem that we solved
- The solution is

$$z(t) = 8\,e^{-kt}$$

Murder Example 3

Solution (cont): From the solution $z(t) = 8\,e^{-kt}$, we have

$$\begin{aligned} z(t) &= T(t) - 22, \quad \text{so} \quad T(t) = z(t) + 22 \\ T(t) &= 22 + 8\,e^{-kt} \end{aligned}$$

- One hour later the body temperature is 28°C

$$T(1) = 28 = 22 + 8\,e^{-k}$$

- Solving

$$6 = 8\,e^{-k} \quad \text{or} \quad e^{k} = \frac{4}{3}$$

- Thus, $k = \ln\left(\frac{4}{3}\right) = 0.2877$

Murder Example

Solution (cont): It only remains to find out when the murder occurred

- At the time of death, t_d, the body temperature is 37°C

$$T(t_d) = 37 = 22 + 8\,e^{-kt_d}$$

- Thus,

$$8\,e^{-kt_d} = 37 - 22 = 15 \quad \text{or} \quad e^{-kt_d} = \frac{15}{8} = 1.875$$

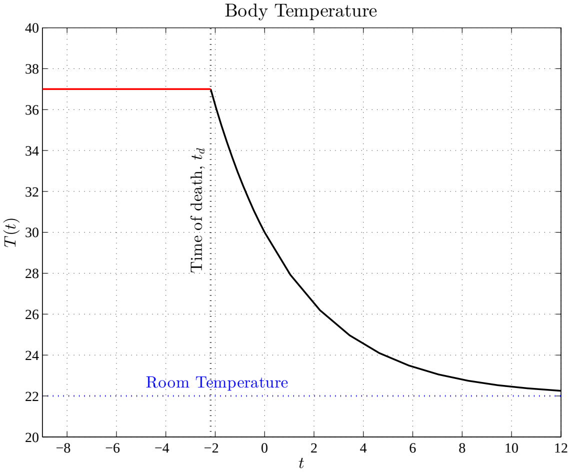
- This gives $-kt_d = \ln(1.875)$ or

$$t_d = -\frac{\ln(1.875)}{k} = -2.19$$

- The murder occurred about 2 hours 11 minutes before the body was found, which places the time of death around **6:19 am**

Murder Example 5

Graph of Body Temperature over time



Solution of General Linear Model 1

Solution of General Linear Model: Consider the Linear Model

$$\frac{dy}{dt} = a\,y + b \qquad \text{with} \qquad y(t_0) = y_0$$

Rewrite equation as

$$\frac{dy}{dt} = a\left(y + \frac{b}{a}\right)$$

Make the substitution $z(t) = y(t) + \frac{b}{a}$, so $\frac{dz}{dt} = \frac{dy}{dt}$ and $z(t_0) = y_0 + \frac{b}{a}$

It follows that

$$\frac{dz}{dt} = a\,z \qquad \text{with} \qquad z(t_0) = y_0 + \frac{b}{a}$$

Solution of General Linear Model 2

The ***linear growth model*** given by

$$\frac{dz}{dt} = a\,z \qquad \text{with} \qquad z(t_0) = y_0 + \tfrac{b}{a},$$

has been solved by our previous method.

The solution is:

$$z(t) = \left(y_0 + \frac{b}{a}\right) e^{a(t-t_0)} = y(t) + \frac{b}{a}.$$

It follows that the solution, $y(t)$ is

$$y(t) = \left(y_0 + \frac{b}{a}\right) e^{a(t-t_0)} - \frac{b}{a}.$$

Solution of General Linear Model

The **linear differential equation** satisfies:

$$\frac{dy}{dt} = a\,y + b = a\left(y + \frac{b}{a}\right)$$

Method (Solution of General Linear Differential Equation)

Consider the ***linear differential equation***

$$\frac{dy}{dt} = a\left(y + \frac{b}{a}\right) \qquad \text{with} \qquad y(t_0) = y_0.$$

With the substitution $z(t) = y(t) + \frac{b}{a}$*, we obtain the solution:*

$$y(t) = \left(y_0 + \frac{b}{a}\right)e^{a(t-t_0)} - \frac{b}{a}.$$

This method produces a ***vertical shift*** of the solution.

Example of Linear Model 1

Example of Linear Model Consider the Linear Model

$$\frac{dy}{dt} = 5 - 0.2\,y \qquad \text{with} \qquad y(3) = 7$$

Rewrite equation as

$$\frac{dy}{dt} = -0.2(y - 25)$$

Make the substitution $z(t) = y(t) - 25$, so $\frac{dz}{dt} = \frac{dy}{dt}$ and $z(3) = -18$

$$\frac{dz}{dt} = -0.2\,z \qquad \text{with} \qquad z(3) = -18$$

Example of Linear Model 2

Example of Linear Model The substituted model is

$$\frac{dz}{dt} = -0.2\,z \qquad \text{with} \qquad z(3) = -18$$

Thus,

$$z(t) = -18\,e^{-0.2(t-3)} = y(t) - 25$$

The solution is

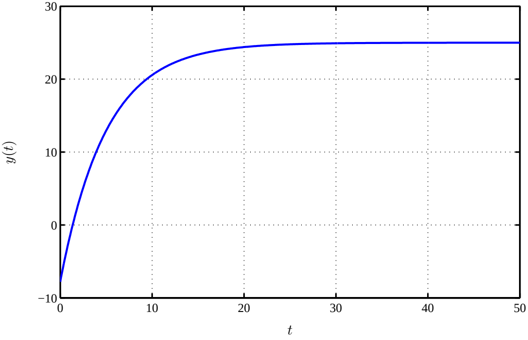
$$y(t) = 25 - 18\,e^{-0.2(t-3)}$$

Example of Linear Model 3

The linear differential equation was transformed into the IVP:

$$\frac{dy}{dt} = -0.2(y - 25), \qquad \text{with} \quad y(3) = 7$$

The graph is given by



Introduction to MatLab

How do we make the previous graph?

MatLab is a powerful software for mathematics, engineering, and the sciences

- MatLab stands for Matrix Laboratory
- Designed for easy managing of vectors, matrices, and graphics
- Valuable subroutines and packages for specialty applications
- It is a necessary tool for anyone in Applied Mathematics
- **Introduction to MatLab**

Autonomous Differential Equation

The general first order differential equation satisfies

$$\frac{dy}{dt} = f(t, y).$$

A very important set of DEs that we study are called **Autonomous Differential Equations**

Definition (Autonomous Differential Equation)

A first order **autonomous** differential equation has the form

$$\frac{dy}{dt} = f(y).$$

The function, f, depends only on the dependent variable.

Qualitative Behavior of Differential Equations

The first step of any **qualitative analysis** is finding **equilibrium solutions**

Definition (Equilibrium Solutions)

Consider **autonomous** DE

$$\frac{dy}{dt} = f(y).$$

If $y(t) = c$ is a constant solution or **equilibrium solution** to this DE, then $\frac{dy}{dt} = 0$. Therefore the constant c is a solution of the algebraic equation

$$f(y) = 0.$$

Equilibrium solutions are also referred to as **fixed points**, **stationary points**, or **critical points**.

Classification of Equilibria

There are a variety of local behaviors near an equilibrium, y_e

1. An **asymptotically stable equilibrium**, often referred to as an **attractor** or **sink** has any nearby solution approach y_e as $t \to \infty$
2. An **unstable equilibrium**, often referred to as a **repeller** or **source** has any nearby solution leave a region about y_e as $t \to \infty$
3. A **neutrally stable equilibrium** has any solution stay nearby the equilibrium, but not approach the equilibrium y_e as $t \to \infty$
4. A **semi-stable equilibrium** (in 1D) has solutions on one side of y_e approach y_e as $t \to \infty$, while solutions on the other side of y_e diverge away from y_e

Taylor's Theorem

Let y_e be an **equilibrium solution** of the DE

$$\frac{dy}{dt} = f(y),$$

so $f(y_e) = 0$.

Theorem (Taylor Series)

If for a range about y_e, the function, f, has infinitely many derivatives at y_e, then $f(y)$ satisfies the ***Taylor Series***

$$f(y) = f(y_e) + f'(y_e)(y - y_e) + \frac{f''(y_e)}{2!}(y - y_e)^2 + \ldots$$

Since $f(y_e) = 0$, then the dominate term near y_e is the linear term $f'(y_e)(y - y_e)$.

Linearization

The next step is finding the **local behavior** near each of the **equilibrium solutions** of the DE

$$\frac{dy}{dt} = f(y).$$

Theorem (Linearization about an Equilibrium Point)

Let y_e be an equilibrium point of the DE above and assume that f has a continuous derivative near y_e.

- *If $f'(y_e) < 0$, then y_e is an asymptotically stable equilibrium.*
- *If $f'(y_e) > 0$, then y_e is an unstable equilibrium.*
- *If $f'(y_e) = 0$, then more information is needed to classify y_e.*

Example: Logistic Growth Model 1

Example: Logistic Growth Model

Consider the logistic growth equation:

$$\frac{dP}{dt} = f(P) = 0.05P\left(1 - \frac{P}{2000}\right)$$

- Equilibria satisfy $f(P_e) = 0$, so
  - $P_e = 0$, the **extinction equilibrium**
  - $P_e = 2000$, the **carrying capacity**
- It is easy to compute $f'(P) = 0.05 - \frac{0.1P}{2000}$
- Since $f'(0) = 0.05 > 0$, $P_e = 0$ is an **unstable equilibrium** or **repeller**
- Since $f'(2000) = -0.05 < 0$, $P_e = 2000$ is a **stable equilibrium** or **attractor**

Example: Logistic Growth Model 2

Geometric Local Analysis: **Equilibria** are $P_e = 0$ and $P_e = 2000$

- The graph of $f(P)$ gives more information
- To the left of $P_e = 0$, $f(P) < 0$
  - Since $\frac{dP}{dt} = f(P) < 0$, $P(t)$ is decreasing
  - Note that this region is outside the region of biological significance
- For $0 < P < 2000$, $f(P) > 0$
  - Since $\frac{dP}{dt} = f(P) > 0$, $P(t)$ is increasing
  - Population monotonically growing in this area
- For $P > 2000$, $f(P) < 0$
  - Since $\frac{dP}{dt} = f(P) < 0$, $P(t)$ is decreasing
  - Population monotonically decreasing in this region

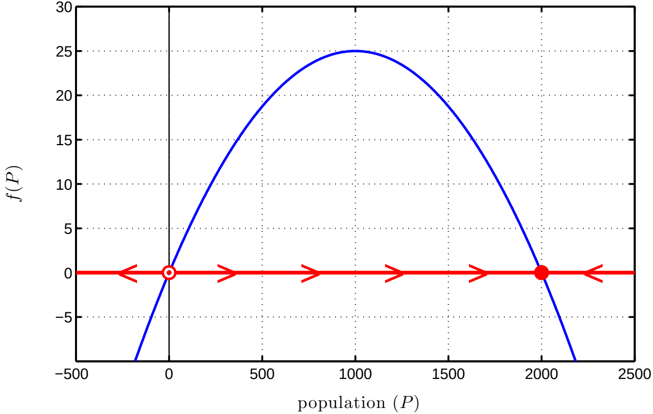
Example: Logistic Growth Model 3

Phase Portrait

- Use the above information to draw a **Phase Portrait** of the behavior of this differential equation along the P-axis
- The behavior of the differential equation is denoted by arrows along the P-axis
  - When $f(P) < 0$, $P(t)$ is decreasing and we draw an **arrow to the left**
  - When $f(P) > 0$, $P(t)$ is increasing and we draw an **arrow to the right**
- **Equilibria**
  - A **solid dot** represents an equilibrium that solutions approach or **stable equilibrium**
  - An **open dot** represents an equilibrium that solutions go away from or **unstable equilibrium**

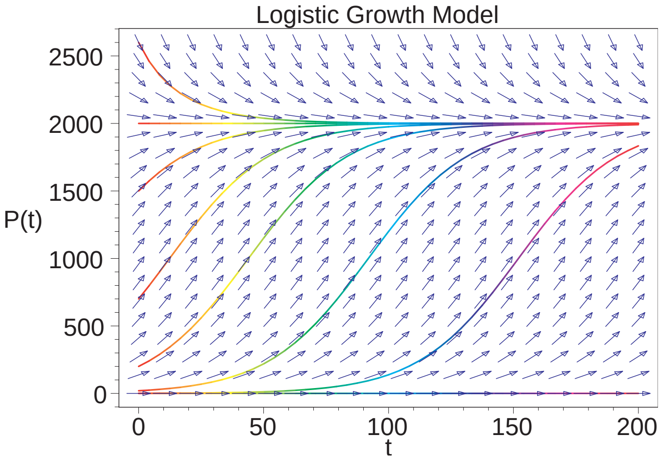
Example: Logistic Growth Model

Phase Portrait: Consists of P-axis, arrows, and equilibria.



Example: Logistic Growth Model

Diagram of Solutions for Logistic Growth Model



Example: Logistic Growth Model

Summary of Qualitative Analysis

- Graph shows solutions either moving away from the equilibrium at $P_e = 0$ or moving toward $P_e = 2000$
- Solutions are increasing most rapidly where $f(P)$ is at a maximum
- Phase portrait shows direction of flow of the solutions without solving the differential equation
- Solutions cannot cross in the tP-plane
- **Phase Portrait analysis**
  - Behavior of a scalar DE found by just graphing function
  - **Equilibria** are **zeros** of function
  - Direction of flow/arrows from sign of function
  - **Stability** of equilibria from whether arrows point toward or away from the equilibria

Example: Sine Function 1

Example: Sine Function

Consider the differential equation:

$$\frac{dx}{dt} = 2\,\sin(\pi x)$$

- Find all equilibria
- Determine the stability of the equilibria
- Sketch the phase portrait
- Show typical solutions

Example: Sine Function 2

For the sine function below:

$$\frac{dx}{dt} = 2\sin(\pi x)$$

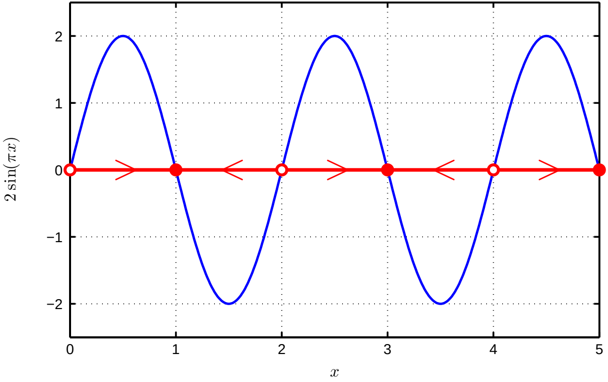
- The equilibria satisfy

$$2\sin(\pi x_e) = 0$$

- Thus, $x_e = n$, where n is any integer
- The sine function passes from negative to positive through $x_e = 0$, so solutions move away from this equilibrium
- The sine function passes from positive to negative through $x_e = 1$, so solutions move toward this equilibrium
- From the function behavior near equilibria
  - All equilibria with $x_e = 2n$ (even integer) are **unstable**
  - All equilibria with $x_e = 2n + 1$ (odd integer) are **stable**

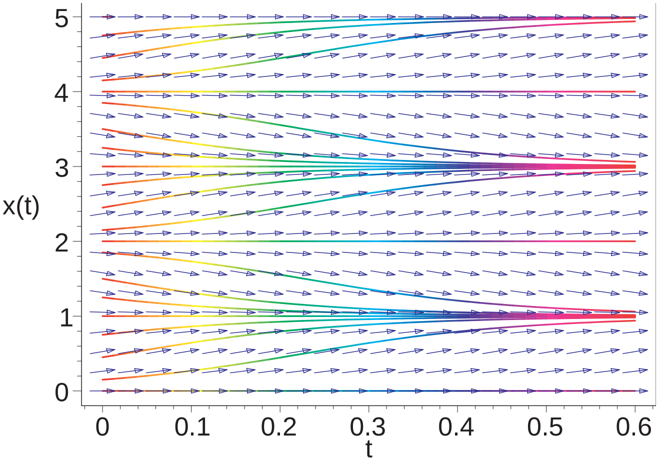
Example: Sine Function

Phase Portrait: Since $2\sin(\pi x)$ alternates sign between integers, the phase portrait follows below:



Example: Sine Function

Diagram of Solutions for Sine Model



Left Snail Model: Introduction

- The shell of a snail exhibits **chirality**, left-handed (sinistral) or right-handed (dextral) coil relative to the central axis
- The Indian conch shell, *Turbinella pyrum*, is primarily a right-handed gastropod [1]
- The left-handed shells are "exceedingly rare"
- The Indians view the rare shells as very holy
  - The Hindu god "Vishnu, in the form of his most celebrated avatar, Krishna, blows this sacred conch shell to call the army of Arjuna into battle"
- So why does nature favor snails with one particular handedness?
- Gould notes that the vast majority of snails grow the dextral form.

[1] S. J. Gould, "Left Snails and Right Minds," *Natural History*, April 1995, **10-18**, and in the compilation *Dinosaur in a Haystack* (1996)

Left Snail Model

1

- Clifford Henry Taubes [2] gives a simple mathematical model to predict the bias of either the dextral or sinistral forms for a given species
  - Assume that the probability of a dextral snail breeding with a sinistral snail is proportional to the product of the number of dextral snails times sinistral snails
  - Assume that two sinistral snails always produce a sinistral snail and two dextral snails produce a dextral snail
  - Assume that a dextral-sinistral pair produce dextral and sinistral offspring with equal probability
- By the first assumption, a dextral snail is twice as likely to choose a dextral snail than a sinistral snail
- Could use real experimental verification of the assumptions

[2] C. H. Taubes, *Modeling Differential Equations in Biology*, Prentice Hall, 2001.

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Left Snail Model 2

Taubes Snail Model

- Let $p(t)$ be the probability that a snail is dextral
- A model that qualitatively exhibits the behavior described on previous slide:

$$\frac{dp}{dt} = \alpha p(1-p)\left(p - \frac{1}{2}\right), \qquad 0 \le p \le 1,$$

where α is some positive constant
- What is the behavior of this differential equation?
- What does its solutions predict about the chirality of populations of snails?

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Left Snail Model

Taubes Snail Model

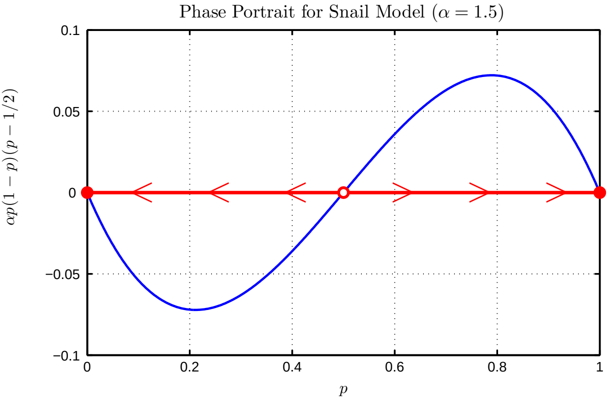
- This differential equation is not easy to solve exactly
- **Qualitative analysis** techniques for this differential equation are relatively easily to show why snails are likely to be in either the dextral or sinistral forms
- The snail model:

$$\frac{dp}{dt} = f(p) = \alpha p(1-p)\left(p - \frac{1}{2}\right), \qquad 0 \le p \le 1,$$

  - **Equilibria** are $p_e = 0, \frac{1}{2}, 1$
  - $f(p) < 0$ for $0 < p < \frac{1}{2}$, so solutions decrease
  - $f(p) > 0$ for $\frac{1}{2} < p < 1$, so solutions increase
  - The equilibrium at $p_e = \frac{1}{2}$ is **unstable**
  - The equilibria at $p_e = 0$ and 1 are **stable**

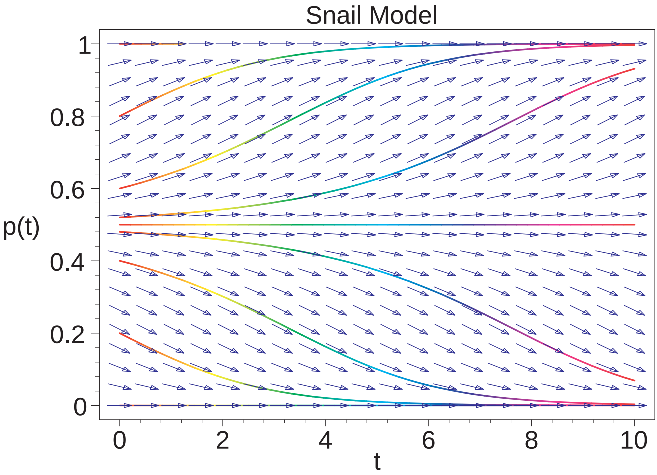
Left Snail Model

Phase Portrait: $$\frac{dp}{dt} = \alpha p(1-p)\left(p - \frac{1}{2}\right)$$



Left Snail Model

Diagram of Solutions for Snail Model



Left Snail Model

Snail Model - Summary

- Figures show the solutions tend toward one of the **stable equilibria**, $p_e = 0$ or 1
- When the solution tends toward $p_e = 0$, then the probability of a dextral snail being found drops to zero, so the population of snails all have the sinistral form
- When the solution tends toward $p_e = 1$, then the population of snails virtually all have the dextral form
- This is what is observed in nature suggesting that this model exhibits the behavior of the evolution of snails
- This does not mean that the model is a good model!
- It simply means that the model exhibits the basic behavior observed experimentally from the biological experiments

Allee Effect 1

Thick-Billed Parrot: *Rhynchopsitta pachyrhycha*

Allee Effect 2

Thick-Billed Parrot: *Rhynchopsitta pachyrhycha*

- A gregarious montane bird that feeds largely on conifer seeds, using its large beak to break open pine cones for the seeds
- These birds used to fly in huge flocks in the mountainous regions of Mexico and Southwestern U. S.
- Largely because of habitat loss, these birds have lost much of their original range and have dropped to only about 1500 breeding pairs in a few large colonies in the mountains of Mexico
- The pressures to log their habitat puts this population at extreme risk for extinction

Allee Effect 3

Thick-Billed Parrot: *Rhynchopsitta pachyrhycha*

- The populations of these birds appear to exhibit a property known in ecology as the **Allee effect**
- These parrots congregate in large social groups for almost all of their activities
- The large group allows the birds many more eyes to watch out for predators
- When the population drops below a certain number, then these birds become easy targets for predators, primarily hawks, which adversely affects their ability to sustain a breeding colony

Allee Effect

Allee Effect:

- Suppose that a population study on thick-billed parrots in a particular region finds that the population, $N(t)$, of the parrots satisfies the differential equation:

$$\frac{dN}{dt} = N\left(r - a(N-b)^2\right),$$

where $r = 0.04$, $a = 10^{-8}$, and $b = 2200$

- Find the equilibria for this differential equation
- Determine the stability of the equilibria
- Draw a phase portrait for the behavior of this model
- Describe what happens to various starting populations of the parrots as predicted by this model

Allee Effect

Equilibria:

- Set the right side of the differential equation equal to zero:

$$N_e\left(r - a(N_e - b)^2\right) = 0$$

- One solution is the **trivial** or **extinction** equilibrium, $N_e = 0$
- When $\left(r - a(N_e - b)^2\right) = 0$, then

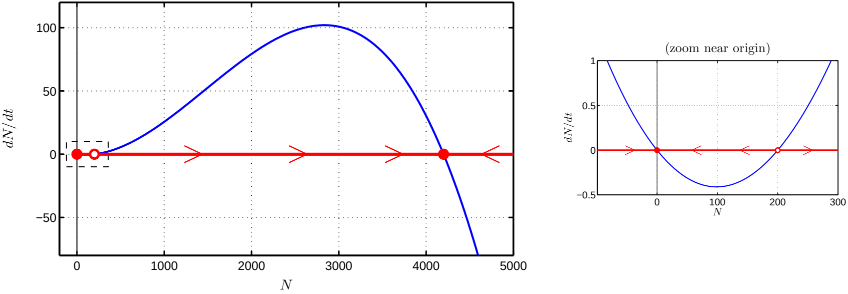
$$(N_e - b)^2 = \frac{r}{a} \qquad \text{or} \qquad N_e = b \pm \sqrt{\frac{r}{a}}$$

- Three distinct equilibria unless $r = 0$ or $b = \sqrt{r/a}$
- With the parameters $r = 0.04$, $a = 10^{-8}$, and $b = 2200$, the equilibria are

$$N_e = 0 \qquad N_e = 200 \qquad 4200$$

Allee Effect

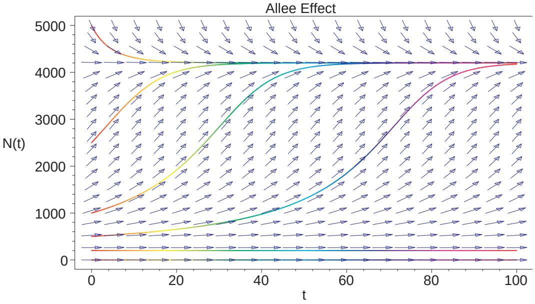
Phase Portrait: Graph of right hand side of differential equation showing equilibria and their stability

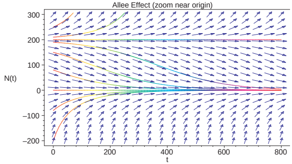


Allee Effect

Solutions: For

$$\frac{dN}{dt} = N\left(r - a(N-b)^2\right)$$





Allee Effect

Interpretation: Model of Allee Effect

- From the phase portrait, the equilibria at 4200 and 0 are stable
- The **threshold** equilibrium at 200 is unstable
  - If the population is above 200, it approaches the **carrying capacity** of this region with the stable population of 4200
  - If the population falls below 200, the model predicts **extinction**, $N_e = 0$
- This agrees with the description for these social birds, which require a critical number of birds to avoid predation
- Below this critical number, the predation increases above reproduction, and the population of parrots goes to extinction
- If the parrot population is larger than 4200, then their numbers will be reduced by starvation (and predation) to the carrying capacity, $N_e = 4200$

Maple Commands for Direction Fields

- $with(DEtools)$:
- $de := diff(P(t), t) = 0.05 \cdot P(t) \cdot \left(1 - \frac{1}{2000} P(t)\right);$
- $DEplot(de,\ P(t),\ t = 0..100,\ P = 0..2500,$ $[[P(0) = 0], [P(0) = 100], [P(0) = 2000], [P(0) = 2500]],\ color = blue,\ linecolor = t);$