Math 337 - Elementary Differential Equations

Lecture Notes – Second Order Linear Equations
Part 2 - Nonhomogeneous

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Outline

Cauchy-Euler Equation

Cauchy-Euler Equation (Also, **Euler Equation**): Consider the differential equation:

$$L[y] = t^2y'' + \alpha ty' + \beta y = 0,$$

where α and β are constants.

Assume $t > 0$ and attempt a solution of the form

$$y(t) = t^r.$$

Note that t^r may not be defined for $t < 0$.

The result is

$$\begin{aligned} L[t^r] &= t^2(r(r-1)t^{r-2}) + \alpha t(rt^{r-1}) + \beta t^r \\ &= t^r[r(r-1) + \alpha r + \beta] = 0. \end{aligned}$$

Thus, obtain quadratic equation

$$F(r) = r(r-1) + \alpha r + \beta = 0.$$

Cauchy-Euler Equation

Cauchy-Euler Equation: The quadratic equation

$$F(r) = r(r-1) + \alpha r + \beta = 0$$

has roots

$$r_1, r_2 = \frac{-(\alpha - 1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}.$$

This is very similar to our **constant coefficient homogeneous** DE.

Real, Distinct Roots: If $F(r) = 0$ has real roots, r_1 and r_2, with $r_1 \neq r_2$, then the **general solution** of

$$L[y] = t^2 y'' + \alpha y' + \beta y = 0,$$

is

$$y(t) = c_1 t^{r_1} + c_2 t^{r_2}, \qquad t > 0.$$

Cauchy-Euler Equation

Example: Consider the equation

$$2t^2y'' + 3ty' - y = 0.$$

By substituting $y(t) = t^r$, we have

$$t^r[2r(r-1) + 3r - 1] = t^r(2r^2 + r - 1) = t^r(2r-1)(r+1) = 0.$$

This has the real roots $r_1 = -1$ and $r_2 = \frac{1}{2}$, giving the **general solution**

$$y(t) = c_1 t^{-1} + c_2\sqrt{t}, \qquad t > 0.$$

Cauchy-Euler Equation

Equal Roots: If $F(r) = (r - r_1)^2 = 0$ has r_1 as a double root, there is one solution, $y_1(t) = t^{r_1}$.

Need a second linearly independent solution.

Note that not only $F(r_1) = 0$, but $F'(r_1) = 0$, so consider

$$\begin{aligned} \frac{\partial}{\partial r} L[t^r] &= \frac{\partial}{\partial r}[t^r F(r)] = \frac{\partial}{\partial r}[t^r (r - r_1)^2] \\ &= (r - r_1)^2 t^r \ln(t) + 2(r - r_1) t^r. \end{aligned}$$

Also,

$$\frac{\partial}{\partial r} L[t^r] = L\left[\frac{\partial}{\partial r}(t^r)\right] = L[t^r \ln(t)].$$

Evaluating these at $r = r_1$ gives

$$L[t^{r_1} \ln(t)] = 0.$$

Cauchy-Euler Equation

Equal Roots: For $F(r) = (r - r_1)^2 = 0$, where r_1 is a double root, then the differential equation

$$L[y] = t^2y'' + \alpha y' + \beta y = 0,$$

was shown to satisfy

$$L[t^{r_1}] = 0 \qquad \text{and} \qquad L[t^{r_1}\ln(t)] = 0.$$

It follows that the **general solution** is

$$y(t) = (c_1 + c_2\ln(t))t^{r_1}.$$

Cauchy-Euler Equation 6

Example: Consider the equation

$$t^2y'' + 5ty' + 4y = 0.$$

By substituting $y(t) = t^r$, we have

$$t^r[r(r-1) + 5r + 4] = t^r(r^2 + 4r + 4) = t^r(r+2)^2 = 0.$$

This only has the real root $r_1 = -2$, which gives **general solution**

$$y(t) = (c_1 + c_2 \ln(t))t^{-2}, \qquad t > 0.$$

Cauchy-Euler Equation

Complex Roots: Assume $F(r) = 0$ has $r = \mu \pm i\nu$ as complex roots, the solutions are still $y(t) = t^r$.

However,

$$t^r = e^{(\mu+i\nu)\ln(t)} = t^{\mu}[\cos(\nu \ln(t)) + i \sin(\nu \ln(t))].$$

As before, we obtain the two linearly independent solutions by taking the real and imaginary parts, so the **general solution** is

$$y(t) = t^{\mu}[c_1 \cos(\nu \ln(t)) + c_2 \sin(\nu \ln(t))].$$

Cauchy-Euler Equation

Example: Consider the equation

$$t^2 y'' + t y' + y = 0.$$

By substituting $y(t) = t^r$, we have

$$t^r[r(r-1) + r + 1] = t^r(r^2 + 1) = 0.$$

This has the complex roots $r = \pm i$ ($\mu = 0$ and $\nu = 1$), which gives the **general solution**

$$y(t) = c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)), \qquad t > 0.$$

Review

Review - Method of undetermined coefficients

- Applicable for constant coefficient nonhomogeneous linear second order differential equations
- The nonhomogeneity is limited to sums and products of:
  - Polynomials
  - Exponentials
  - Sines and Cosines
- Solutions reduce to solving linear equations in the unknown coefficients

Variation of Parameters

Variation of Parameters - This method provides a more general method to solve nonhomogeneous problems

- Technique again begins with a **fundamental set of solutions** to the **homogeneous problem**
- Fundamental set allows creation of the **Wronskian**
- Obtain integral formulation from the fundamental solution with the nonhomogeneous function
- General solution is again formulated from a **particular solution** added to the **homogeneous solution**

Motivating Example 1

Motivating Example: Consider the nonhomogeneous problem

$$y'' + 4y = 3\csc(t),$$

which is inappropriate for the **Method of Undetermined Coefficients**

The **homogeneous solution** is

$$y_c(t) = c_1\cos(2t) + c_2\sin(2t)$$

Generalize this solution to the form

$$y(t) = u_1(t)\cos(2t) + u_2(t)\sin(2t),$$

where the functions u_1 and u_2 are to be determined

Differentiate

$$y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t) + u_1'(t)\cos(2t) + u_2'(t)\sin(2t)$$

Motivating Example 2

Motivating Example: The **general solution** has the form

$$y(t) = u_1(t)\cos(2t) + u_2(t)\sin(2t)$$

Since there is one general solution, there must be a condition relating u_1 and u_2

The computations are simplified by taking the relationship

$$u_1'(t)\cos(2t) + u_2'(t)\sin(2t) = 0$$

This simplifies the derivative of the general solution to

$$y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t)$$

Differentiating again yields:

$$y''(t) = -4u_1(t)\cos(2t) - 4u_2(t)\sin(2t) - 2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t)$$

Motivating Example 3

Motivating Example: The differential equation is

$$y'' + 4y = 3\csc(t),$$

so substituting the general solution gives

$$\begin{aligned} -4u_1(t)\cos(2t) - 4u_2(t)\sin(2t) - 2u_1'(t)\sin(2t) & \\ +2u_2'(t)\cos(2t) + 4(u_1(t)\cos(2t) + u_2(t)\sin(2t)) &= 3\csc(t), \end{aligned}$$

which simplifies to

$$-2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) = 3\csc(t)$$

This equation is combined with our earlier simplifying condition

$$u_1'(t)\cos(2t) + u_2'(t)\sin(2t) = 0$$

Motivating Example 4

Motivating Example: The previous equations give two **linear algebraic equations** in u_1' and u_2'

$$\begin{aligned} u_1'(t)\cos(2t) + u_2'(t)\sin(2t) &= 0 \\ -2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) &= 3\csc(t) \end{aligned}$$

The first equation gives

$$u_2'(t) = -u_1'(t)\cot(2t)$$

It follows that (with trig identities)

$$u_1'(t) = -\frac{3}{2}\csc(t)\sin(2t) = -3\cos(t)$$

and (with trig identities)

$$u_2'(t) = 3\cos(t)\cot(2t) = \frac{3}{2}\csc(t) - 3\sin(t)$$

Motivating Example 5

Motivating Example: We solve the equations for u_1' and u_2'

$$u_1'(t) = -3\cos(t),$$

so

$$u_1(t) = -3\sin(t) + c_1$$

Simlarly,

$$u_2'(t) = \tfrac{3}{2}\csc(t) - 3\sin(t),$$

so

$$u_2(t) = \tfrac{3}{2}\ln|\csc(t) - \cot(t)| + 3\cos(t) + c_2$$

It follows that the general solution is

$$\begin{aligned} y(t) &= u_1(t)\cos(2t) + u_2(t)\sin(2t) \\ &= -3\sin(t)\cos(2t) + \tfrac{3}{2}\sin(2t)\ln|\csc(t) - \cot(t)| + 3\cos(t)\sin(2t) \\ &\quad +c_1\cos(2t) + c_2\sin(2t), \end{aligned}$$

which shows the **homogeneous** and **particular** solutions

Variation of Parameters 1

Technique of Variation of Parameters: Consider the nonhomogeneous problem

$$y'' + p(t)y' + q(t)y = g(t),$$

where p, q, and g are given continuous functions

Assume we know the **homogeneous solution**:

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

Try a **general solution** of the form

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where the functions u_1 and u_2 are to be determined

Differentiating yields

$$y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t) + u_1'(t)y_1(t) + u_2'(t)y_2(t)$$

Variation of Parameters 2

Variation of Parameters: As before, there must be a condition relating u_1 and u_2, so take

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$

This simplifies the derivative of the general solution to

$$y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t)$$

Differentiating again yields:

$$y''(t) = u_1(t)y_1''(t) + u_2(t)y_2''(t) + u_1'(t)y_1'(t) + u_2'(t)y_2'(t)$$

Variation of Parameters

Variation of Parameters: We now have expressions for the **general solution**, $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, and its derivatives, $y'(t)$ and $y''(t)$, which we substitute into the nonhomogeneous problem:

$$y'' + p(t)y' + q(t)y = g(t),$$

This can be written in the form:

$$\begin{aligned} u_1(t)\left[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)\right] \\ + u_2(t)\left[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)\right] \\ + u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t) \end{aligned}$$

The quantities in the square brackets are **zero**, since y_1 and y_2 are solutions of the **homogeneous equation**, leaving

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$

Variation of Parameters

Variation of Parameters: This gives two **linear algebraic equations** in u_1' and u_2'

$$\begin{aligned} u_1'(t)y_1(t) + u_2'(t)y_2(t) &= 0 \\ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) &= g(t) \end{aligned}$$

Recall Cramer's Rule for solving a system of two linear equations in two unknowns, which above are the functions $u_1'(t)$ and $u_2'(t)$.

$$u_1'(t) = \frac{\det \begin{vmatrix} 0 & y_2(t) \\ g(t) & y_2'(t) \end{vmatrix}}{\det \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} \qquad \text{and} \qquad u_2'(t) = \frac{\det \begin{vmatrix} y_1(t) & 0 \\ y_2'(t) & g(t) \end{vmatrix}}{\det \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}}$$

From before we recognize the denominator as the **Wronskian**:

$$W[y_1, y_2](t) = \det \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

Variation of Parameters

Variation of Parameters: It follows that

$$u_1'(t) = \frac{\det \begin{vmatrix} 0 & y_2(t) \\ g(t) & y_2'(t) \end{vmatrix}}{W[y_1, y_2](t)} \qquad \text{and} \qquad u_2'(t) = \frac{\det \begin{vmatrix} y_1(t) & 0 \\ y_2'(t) & g(t) \end{vmatrix}}{W[y_1, y_2](t)}$$

Solving this, we obtain:

$$u_1'(t) = -\frac{y_2(t)g(t)}{W[y_1, y_2](t)} \qquad \text{and} \qquad u_2'(t) = \frac{y_1(t)g(t)}{W[y_1, y_2](t)},$$

which can be integrated.

Variation of Parameters

Variation of Parameters: The equations for u_1' and u_2' are integrated yielding

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1, y_2](t)}dt + c_1$$

and

$$u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1, y_2](t)}dt + c_2$$

If these integrals can be evaluated, then the **general solution** can be written

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

Otherwise, the solution is given in its integral form

Variation of Parameters Theorem

Consider the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t),$$

Theorem

If the functions p, q, and g are continuous on an open interval I, and if y_1 and y_2 form a **fundamental set of solutions** *of the homogeneous equation. Then a* **particular solution** *of the nonhomogeneous problem is*

$$y_p(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1, y_2](s)} ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1, y_2](s)} ds,$$

where $t_0 \in I$. The **general solution** *is*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t).$$

Variation of Parameters Example 1

Example: Solve the differential equation

$$t^2y'' - 2y = 3t^2 - 1, \qquad t > 0$$

As always, first solve the homogeneous equation:

$$t^2y'' - 2y = 0,$$

which is a **Cauchy-Euler Equation**

Attempt solution $y(t) = t^r$, giving

$$t^r[r(r-1) - 2] = t^r(r^2 - r - 2) = t^r(t-2)(t+1) = 0$$

This gives the **homogeneous** solution

$$y_c(t) = c_1y_1(t) + c_2y_2(t) = c_1t^{-1} + c_2t^2$$

Variation of Parameters Example 2

Example: The differential equation

$$t^2 y'' - 2y = 3t^2 - 1, \qquad t > 0$$

has **homogeneous** solutions $y_1(t) = t^{-1}$ and $y_2(t) = t^2$

Compute the **Wronskian**

$$W[t^{-1}, t^2](t) = \det \begin{vmatrix} t^{-1} & t^2 \\ -t^{-2} & 2t \end{vmatrix} = 3$$

To use the **Variation of Parameters**, we rewrite the DE

$$y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2} = g(t), \qquad t > 0,$$

Variation of Parameters Example 3

Example: From the theorem above, a **particular solution** satisfies

$$\begin{aligned} y_p(t) &= -y_1(t)\int_{t_0}^{t}\frac{y_2(s)g(s)}{W[y_1,y_2](s)}ds + y_2(t)\int_{t_0}^{t}\frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds, \\ &= -t^{-1}\int^{t}\frac{s^2(3-s^{-2})}{3}ds + t^2\int^{t}\frac{s^{-1}(3-s^{-2})}{3}ds \\ &= -t^{-1}\left(\frac{t^3}{3}-\frac{t}{3}\right) + t^2\left(\ln(t)+\frac{1}{6}t^{-2}\right) \\ &= -\frac{t^2}{3}+\frac{1}{3}+t^2\ln(t)+\frac{1}{6} \end{aligned}$$

Since the first term is part of the homogeneous solution, we write the **general solution** as

$$y(t) = c_1t^{-1} + c_2t^2 + \frac{1}{2} + t^2\ln(t), \qquad t > 0$$