Math 337 - Elementary Differential Equations

Lecture Notes – Second Order Linear Equations

Joseph M. Mahaffy,
⟨mahaffy@math.sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://www-rohan.sdsu.edu/~jmahaffy

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Outline

Introduction

Introduction

- Introduction to second order differential equations
- Linear Theory and Fundamental sets of solutions
- Homogeneous linear second order differential equations
- Nonhomogeneous linear second order differential equations
  - Method of undetermined coefficients
  - Variation of parameters
  - Reduction of order

Second Order DE

Second Order Differential Equation with an independent variable y, dependent variable t, and prescribed function, f:

$$y'' = f(t, y, y'),$$

- Often arises in physical problems, *e.g.*, Newton's Law where force depends on acceleration
- **Solution** is a twice continuously differentiable function
- **Initial value problem** requires two initial conditions

$$y(t_0) = y_0 \qquad \text{and} \qquad y'(t_0) = y_1$$

- Can develop **Existence and Uniqueness** conditions

Linear Second Order DE

Linear Second Order Differential Equation:

$$y'' + p(t)y' + q(t)y = g(t)$$

- Equation is **homogeneous** if $g(t) = 0$ for all t
- Otherwise, **nonhomogeneous**
- Equation is **constant coefficient** equation if written

$$ay'' + by' + cy = g(t),$$

where $a \neq 0$, b, and c are constants

Dynamical system formulation

Dynamical system formulation Suppose

$$y'' = f(t, y, y')$$

and introduce variables $x_1 = y$ and $x_2 = y'$

Obtain **dynamical system**

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(t, x_1, x_2) \end{aligned}$$

The **state variables** are y and y', which have solutions producing **trajectories** or **orbits** in the **phase plane**

For movement of a particle, one can think of the DE governing the dynamics produces by Newton's Law of motion and the **phase plane orbits** show the **position** and **velocity** of the particle

Classic Examples

- **Spring Problem** with mass m position $y(t)$, k spring constant, γ viscous damping, and external force $F(t)$
  - Unforced, undamped oscillator, $my'' + ky = 0$
  - Unforced, damped oscillator, $my'' + \gamma y' + ky = 0$
  - Forced, undamped oscillator, $my'' + ky = F(t)$
  - Forced, damped oscillator, $my'' + \gamma y' + ky = F(t)$
- **Pendulum Problem**- mass m, drag c, length L, $\gamma = \frac{c}{mL}$, $\omega^2 = \frac{g}{L}$, angle $\theta(t)$
  - Nonlinear, $\theta'' + \gamma\theta' + \omega^2 \sin(\theta) = 0$
  - Linearized, $\theta'' + \gamma\theta' + \omega^2\theta = 0$
- **RLC Circuit**
  - Let R be the resistance (ohms), C be capacitance (farads), L be inductance (henries), $e(t)$ be impressed voltage
  - Kirchhoff's Law for $q(t)$, charge on the capacitor

$$Lq'' + Rq' + \frac{q}{C} = e(t),$$

Existence and Uniqueness

Theorem (Existence and Uniqueness)

Let $p(t)$, $q(t)$, and $g(t)$ be continuous on an open interval I, let $t_0 \in I$, and let y_0 and y_1 be given numbers. Then there exists a unique solution $y = \phi(t)$ of the 2^{nd} order differential equation:

$$y'' + p(t)y' + q(t)y = g(t),$$

that satisfies the initial conditions

$$y(t_0) = y_0 \qquad \text{and} \qquad y'(t_0) = y_1.$$

This unique solution exists throughout the interval I.

Linear Operator

Theorem (Linear Differential Operator)

Let L satisfy $L[y] = y'' + py' + qy$, where p and q are continuous functions on an interval I. If y_1 and y_2 are twice continuously differentiable functions on I and c_1 and c_2 are constants, then

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2].$$

Proof uses linearity of differentiation.

Theorem (Principle of Superposition)

*Let $L[y] = y'' + py' + qy$, where p and q are continuous functions on an interval I. If y_1 and y_2 are two solutions of $L[y] = 0$ (**homogeneous equation**), then the linear combination*

$$y = c_1y_1 + c_2y_2$$

is also a solution for any constants c_1 and c_2.

Wronskian

Wronskian: Consider the **linear homogeneous 2^{nd} order DE**

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

with $p(t)$ and $q(t)$ continuous on an interval I

Let y_1 and y_2 be solutions satisfying $L[y_i] = 0$ for $i = 1, 2$ and define the **Wronskian** by

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{vmatrix} = y_1(t)y'_2(t) - y'_1(t)y_2(t).$$

If $W[y_1, y_2](t) \neq 0$ on I, then the general solution of $L[y] = 0$ satisfies

$$y(t) = c_1y_1(t) + c_2y_2(t).$$

Fundamental Set of Solutions

Theorem

Let y_1 and y_2 be two solutions of

$$y'' + p(t)y' + q(t)y = 0,$$

*and assume the Wronskian, $W[y_1, y_2](t) \neq 0$ on I. Then y_1 and y_2 form a **fundamental set of solutions**, and the general solution is given by*

$$y(t) = c_1y_1(t) + c_2y_2(t).$$

where c_1 and c_2 are arbitrary constants. If there are given initial conditions, $y(t_0) = y_0$ and $y'(t_0) = y_1$ for some $t_0 \in I$, then these conditions determine c_1 and c_2 uniquely.

Homogeneous Equations 1

Homogeneous Equation: The general 2^{nd} order constant coefficient homogeneous differential equation is written:

$$ay'' + by' + cy = 0$$

This can be written as a **system of 1^{st} order differential equations**

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}\mathbf{x},$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

This has a the general solution

$$\mathbf{x} = c_1\begin{pmatrix} y_1 \\ y'_1 \end{pmatrix} + c_2\begin{pmatrix} y_2 \\ y'_2 \end{pmatrix}$$

Homogeneous Equations 2

Characteristic Equation: Obtain **characteristic equation** by solving

$$\det|\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} -\lambda & 1 \\ -c/a & -b/a-\lambda \end{vmatrix} = \frac{1}{a}\left(a\lambda^2+b\lambda+c\right)=0$$

Find eigenvectors by solving

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v} = \begin{pmatrix} -\lambda & 1 \\ -c/a & -b/a-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If λ is an eigenvalue, then it follows the corresponding eigenvector is

$$\mathbf{v} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

Then a solution is given by

$$\mathbf{x} = e^{\lambda t}\mathbf{v} = \begin{pmatrix} e^{\lambda t} \\ \lambda e^{\lambda t} \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

Homogeneous Equations 3

Theorem

*Let λ_1 and λ_2 be the roots of the **characteristic equation***

$$a\lambda^2+b\lambda+c=0.$$

*Then the general solution of the **homogeneous DE**,*

$$ay''+by'+cy=0,$$

satisfies

$$y(t)=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t} \qquad \text{if } \lambda_1\neq\lambda_2 \text{ are real,}$$
$$y(t)=c_1e^{\lambda_1 t}+c_2te^{\lambda_1 t} \qquad \text{if } \lambda_1=\lambda_2,$$
$$y(t)=c_1e^{\mu t}\cos(\nu t)+c_2e^{\mu t}\sin(\nu t) \qquad \text{if } \lambda_{1,2}=\mu\pm i\nu \text{ are complex.}$$

Homogeneous Equations - Example

Consider the IVP

$$y''+5y'+6y=0, \qquad y(0)=2, \quad y'(0)=3.$$

The **characteristic equation** is $\lambda^2+5\lambda+6=(\lambda+3)(\lambda+2)=0$, so $\lambda=-3$ and $\lambda=-2$

The general solution is $y(t)=c_1e^{-3t}+c_2e^{-2t}$

From the initial conditions

$$y(0)=c_1+c_2=2 \qquad \text{and} \qquad y\ (0)=-3c_1-2c_2=3$$

When solved simultaneously, gives $c_1=-7$ and $c_2=9$, so

$$y(t)=9e^{-2t}-7e^{-3t}$$

This problem is the same as solving

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0)=\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Nonhomogeneous Equations 1

Nonhomogeneous Equations: Consider the DE

$$L[y]=y''+p(t)y'+q(t)y=g(t)$$

Theorem

*Let y_1 and y_2 form a fundamental set of solutions to the **homogeneous equation**, $L[y]=0$. Also, assume that Y_p is a* **particular solution** *to $L[Y_p]=g(t)$. Then the general solution to $L[Y]=g(t)$ is given by:*

$$y(t)=c_1y_1(t)+c_2y_2(t)+Y_p(t).$$

Nonhomogeneous Equations 2

The previous theorem provides the **basic solution strategy** for **2^{nd} order nonhomogeneous differential equations**

- Find the general solution $c_1y_1(t) + c_2y_2(t)$ of the homogeneous equation
  - This is sometimes called the **complementary solution** and often denoted $y_c(t)$ or $y_h(t)$
- Find any solution of the **nonhomogeneous DE**
  - This is usually called the **particular solution** and often denoted $y_p(t)$
- Add these solutions together for the **general solution**
- Two common methods for obtaining the particular solution
  - For common specific functions and constant coefficients for the DE, use the **method of undetermined coefficients**
  - More general method uses **method of variation of parameters**

Method of Undetermined Coefficients

Method of Undetermined Coefficients - Example 1: Consider the DE

$$y'' - 3y' - 4y = 3e^{2t}$$

The characteristic equation is $\lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4) = 0$, so the homogeneous solution is

$$y_c(t) = c_1e^{-t} + c_2e^{4t}$$

Neither solution matches the **forcing function**, so try

$$y_p(t) = Ae^{2t}$$

It follows that

$$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = -6Ae^{2t} = 3e^{2t} \quad \text{or} \quad A = -\frac{1}{2}$$

The solution combines these to obtain

$$y(t) = c_1e^{-t} + c_2e^{4t} - \frac{1}{2}e^{2t}$$

Method of Undetermined Coefficients

Method of Undetermined Coefficients - Example 2: Consider

$$y'' - 3y' - 4y = 5\sin(t)$$

From before, the homogeneous solution is $y_c(t) = c_1e^{-t} + c_2e^{4t}$

Neither solution matches the **forcing function**, so try

$$\begin{aligned} y_p(t) &= A\sin(t) + B\cos(t) \qquad \text{so} \\ y_p'(t) &= A\cos(t) - B\sin(t) \quad \text{and} \quad y_p''(t) = -A\sin(t) - B\cos(t) \end{aligned}$$

It follows that

$$(-A + 3B - 4A)\sin(t) + (-B - 3A - 4B)\cos(t) = 5\sin(t)$$

or $3A + 5B = 0$ and $3B - 5A = 5$ or $A = -\frac{25}{34}$ and $B = \frac{15}{34}$

The solution combines these to obtain

$$y(t) = c_1e^{-t} + c_2e^{4t} + \frac{15}{34}\cos(t) - \frac{25}{34}\sin(t)$$

Method of Undetermined Coefficients

Method of Undetermined Coefficients - Example 3: Consider

$$y'' - 3y' - 4y = 2t^2 - 7$$

From before, the homogeneous solution is $y_c(t) = c_1e^{-t} + c_2e^{4t}$

Neither solution matches the **forcing function**, so try

$$y_p(t) = At^2 + Bt + C$$

It follows that

$$2A - 3(2At + B) - 4(At^2 + Bt + C) = 2t^2 - 7,$$

so matching coefficients gives $-4A = 2$, $-6A - 4B = 0$, and $2A - 3B - 4C = -7$, which yields $A = -\frac{1}{2}$, $B = \frac{3}{4}$ and $C = \frac{15}{16}$

The solution combines these to obtain

$$y(t) = c_1e^{-t} + c_2e^{4t} - \frac{t^2}{2} + \frac{3t}{4} + \frac{15}{16}$$

Method of Undetermined Coefficients

Superposition Principle: Suppose that $g(t) = g_1(t) + g_2(t)$. Also, assume that $y_{1p}(t)$ and $y_{2p}(t)$ are **particular solutions** of

$$\begin{aligned} ay'' + by' + cy &= g_1(t) \\ ay'' + by' + cy &= g_2(t), \end{aligned}$$

respectively.

Then $y_{1p}(t) + y_{2p}(t)$ is a solution of

$$ay'' + by' + cy = g(t)$$

From our previous examples, the solution of

$$y'' - 3y' - 4y = 3e^{2t} + 5\sin(t) + 2t^2 - 7$$

satisfies

$$y(t) = c_1e^{-t} + c_2e^{4t} - \frac{1}{2}e^{2t} + \frac{15}{34}\cos(t) - \frac{25}{34}\sin(t) - \frac{t^2}{2} + \frac{3t}{4} + \frac{15}{16}$$

Method of Undetermined Coefficients

Method of Undetermined Coefficients - Example 4: Consider

$$y'' - 3y' - 4y = 5e^{-t}$$

From before, the homogeneous solution is $y_c(t) = c_1e^{-t} + c_2e^{4t}$

Since the **forcing function** matches one of the solutions in $y_c(t)$, we attempt a particular solution of the form

$$y_p(t) = Ate^{-t},$$

so

$$y_p'(t) = A(1-t)e^{-t} \qquad \text{and} \qquad y_p''(t) = A(t-2)e^{-t}$$

It follows that

$$\left(A(t-2) - 3A(1-t) - 4At\right)e^{-t} = -5Ae^{-t} = 5e^{-t},$$

Thus, $A = -1$

The solution combines these to obtain

$$y(t) = c_1e^{-t} + c_2e^{4t} - te^{-t}$$

Method of Undetermined Coefficients

Method of Undetermined Coefficients: Consider the problem

$$ay'' + by' + cy = g(t)$$

- First solve the **homogeneous equation**, which must have constant coefficients
- The **nonhomogeneous function**, $g(t)$, must be in the class of functions with polynomials, exponentials, sines, cosines, and products of these functions
- $g(t) = g_1(t) + ... + g_n(t)$ is a sum the type of functions listed above
- Find **particular solutions**, $y_{ip}(t)$, for each $g_i(t)$
- General solution combines the homogeneous solution with all the particular solutions
- The arbitrary constants with the homogeneous solution are found to satisfy initial conditions for unique solution

Method of Undetermined Coefficients

Summary Table for Method of Undetermined Coefficients
The table below shows how to choose a particular solution

Particular solution for $ay'' + by' + cy = g(t)$

$g(t)$	$y_p(t)$
$P_n(t) = a_nt^n + ... + a_1t + a_0$	$t^s\left(A_nt^n + ... + A_1t + A_0\right)$
$P_n(t)e^{\alpha t}$	$t^s\left(A_nt^n + ... + A_1t + A_0\right)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin(\beta t)\\ \cos(\beta t)\end{cases}$	$t^s\left[\left(A_nt^n + ... + A_1t + A_0\right)e^{\alpha t}\cos(\beta t) + \left(B_nt^n + ... + B_1t + B_0\right)e^{\alpha t}\sin(\beta t)\right]$

Note: The s is the smallest integer ($s = 0, 1, 2$) that ensures no term in $y_p(t)$ is a solution of the homogeneous equation

Forced Vibrations

Forced Vibrations: The damped spring-mass system with an external force satisfies the equation:

$$my'' + \gamma y' + ky = F(t)$$

Example 1

- Assume a 2 kg mass and that a 4 N force is required to maintain the spring stretched 0.2 m
- Suppose that there is a damping coefficient of $\gamma = 4$ kg/sec
- Assume that an external force, $F(t) = 0.5\sin(4t)$ is applied to this spring-mass system
- The mass begins at rest, so $y(0) = y'(0) = 0$
- Set up and solve this system

Example 1 (1)

Example 1: The first condition allows computation of the spring constant, k

Since a 4 N force is required to maintain the spring stretched 0.2 m,

$$k(0.2) = 4 \qquad \text{or} \qquad k = 20$$

It follows that the damped spring-mass system described in this problem satisfies:

$$2y'' + 4y' + 20y = 0.5\sin(4t)$$

or equivalently

$$y'' + 2y' + 10y = 0.25\sin(4t), \qquad \text{with} \quad y(0) = y'(0) = 0$$

Example 1 (2)

Solution: Apply the **Method of Undetermined Coefficients** to

$$y'' + 2y' + 10y = 0.25\sin(4t)$$

The **Homogeneous Solution:**

The characteristic equation is $\lambda^2 + 2\lambda + 10 = 0$, which has solution $\lambda = -1 \pm 3i$, so the homogeneous solution is

$$y_c(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)$$

The **Particular Solution:**

Guess a solution of the form:

$$y_p(t) = A\cos(4t) + B\sin(4t)$$

Example 1 (3)

Solution: Want $y_p'' + 2y_p' + 10y_p = 0.25\sin(4t)$, so with $y_p(t) = A\cos(4t) + B\sin(4t)$

$$\begin{aligned} -16A\cos(4t) - 16B\sin(4t) + 2(-4A\sin(4t) + 4B\cos(4t)) & \\ +10(A\cos(4t) + B\sin(4t)) &= 0.25\sin(4t) \end{aligned}$$

Equating the coefficients of the sine and cosine terms gives:

$$\begin{aligned} -6A + 8B &= 0, \\ -8A - 6B &= 0.25, \end{aligned}$$

which gives $A = -\frac{1}{50}$ and $B = -\frac{3}{200}$

The solution is

$$y(t) = e^{-t}\left(c_1\cos(3t) + c_2\sin(3t)\right) - \frac{1}{50}\cos(4t) - \frac{3}{200}\sin(4t)$$

Example 1 4

Solution: With the solution

$$y(t) = e^{-t}\left(c_1\cos(3t) + c_2\sin(3t)\right) - \frac{1}{50}\cos(4t) - \frac{3}{200}\sin(4t),$$

we apply the initial conditions.

$$y(0) = 0 = c_1 - \frac{1}{50} \qquad \text{or} \qquad c_1 = \frac{1}{50}$$

$$y'(0) = 3c_2 - c_1 - \frac{3}{50} = 0 \qquad \text{or} \qquad c_2 = \frac{2}{75}$$

The solution to this spring-mass problem is

$$y(t) = e^{-t}\left(\frac{1}{50}\cos(3t) + \frac{2}{75}\sin(3t)\right) - \frac{1}{50}\cos(4t) - \frac{3}{200}\sin(4t)$$

Frequency Response 1

Frequency Response: Rewrite the **damped spring-mass system**:

$$y'' + 2\delta y' + \omega_0^2 y = f(t),$$

with $\omega_0^2 = k/m$ and $\delta = \gamma/(2m)$

Example 2: Let $f(t) = K\cos(\omega t)$ and find a particular solution to this equation

Take

$$y_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

Upon differentiation and collecting cosine terms, we have

$$-A\omega^2 + 2B\delta\omega + A\omega_0^2 = K$$

The sine terms satisfy

$$-B\omega^2 - 2A\delta\omega + B\omega_0^2 = 0$$

Frequency Response 2

Frequency Response: Coefficient from our Undetermined Coefficient method give the linear system

$$\begin{aligned}(\omega_0^2 - \omega^2)A + 2\delta\omega B &= K, \\ -2\delta\omega A + (\omega_0^2 - \omega^2)B &= 0.\end{aligned}$$

This has the solution

$$A = \frac{K(\omega_0^2 - \omega^2)}{((\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2)} \qquad \text{and} \qquad B = \frac{2K\delta\omega}{((\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2)}$$

It follows that the **particular solution** is

$$y_p(t) = \frac{K\left[(\omega_0^2 - \omega^2)\cos(\omega t) + 2\delta\omega\sin(\omega t)\right]}{((\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2)}$$

Frequency Response 3

Frequency Response: The model

$$y'' + 2\delta y' + \omega_0^2 y = K\cos(\omega t),$$

has exponentially decaying solutions from the **homogeneous solution**.

Thus, the solution approaches the **particular solution**

$$y_p(t) = \frac{K\left[(\omega_0^2 - \omega^2)\cos(\omega t) + 2\delta\omega\sin(\omega t)\right]}{((\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2)}$$

This **particular solution** has a **maximum response** when $\omega = \omega_0$

Thus, **tuning** the forcing function to the **natural frequency**, ω_0 yields the maximum response