Math 337 - Elementary Differential Equations

Lecture Notes – Power Series
Ordinary Point

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Outline

1. Introduction
   - Example
   - Review Power Series
2. Series Solutions of Differential Equations
   - Airy's Equation
   - Chebyshev's Equation

Introduction

Introduction - Solving 2^{nd} **order differential equations**

$$P(t)y'' + Q(t)y' + R(t)y = g(t)$$

- **Constant coefficient** - $P(t)$, $Q(t)$, and $R(t)$ are constant
  - **Homogeneous** - Solutions $y(t) = ce^{\lambda t}$
  - Create **2D system** of 1^{st} order differential equations
  - **Nonhomogeneous** - **Method of Undetermined Coefficients**
  - **Laplace transforms**
- **Nonconstant coefficient** - $P(t)$, $Q(t)$, and $R(t)$
  - **Cauchy-Euler equations**
  - **Nonhomogeneous** - **Variation of Parameters**
  - Now learn **Power Series** methods

Example

Example: Consider the 2^{nd} **order differential equation**

$$y'' - y = 0,$$

which is easily solved by earlier methods

Instead of trying a solution $y(t) = ce^{\lambda t}$, try

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

It readily follows that

$$y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} \qquad \text{and} \qquad y''(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

Note that the lower index of the sums increases, as the derivative on a constant is **zero**

Example

Example: With $y'' - y = 0$, substitute the **Power Series** solution

$$y(t) = \sum_{n=0}^{\infty} a_n t^n,$$

which gives

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=0}^{\infty} a_n t^n = 0$$

- **Important observations:**
  - **Index of the sums** differs where it starts
  - **Powers** of t are different
  - The **homogeneous** term has the **power series**

$$0 = \sum_{n=0}^{\infty} b_n t^n, \quad \text{where} \quad b_n = 0 \quad \text{for all } n$$

Example

Example: Let $k = n - 2$, then we can rewrite the sum for $y''(t)$ as

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}t^k$$

However, the indices of a sum are dummy variables, so exchange k back to n

The differential equation can be written:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n - \sum_{n=0}^{\infty} a_n t^n = 0,$$

which when combined gives

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right] t^n = 0$$

Example

Example: Since

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right] t^n = 0,$$

it follows that

$$(n+2)(n+1)a_{n+2} - a_n = 0$$

The first two coefficients, a_0 and a_1 are **arbitrary**, then all other coefficients are specified by the **recursive relation**:

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)}$$

Thus, with a_0 arbitrary

$$a_2 = \frac{a_0}{2!}, \quad a_4 = \frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}, \quad \dots, \quad a_{2n} = \frac{a_0}{(2n)!}$$

Example

Example: Similarly, with a_1 arbitrary

$$a_3 = \frac{a_1}{3 \cdot 2}, \quad a_5 = \frac{a_3}{5 \cdot 4} = \frac{a_1}{5!}, \quad \dots, \quad a_{2n+1} = \frac{a_1}{(2n+1)!}$$

It follows that we have two **linearly independent** solutions

$$y_1(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \quad \text{and} \quad y_2(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!},$$

with the **general solution**

$$y(t) = a_0 y_1(t) + a_1 y_2(t)$$

Note: $y_1(t) = \cosh(t)$ and $y_2(t) = \sinh(t)$

Review Power Series

Review Power Series: Consider the power series:

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n$$

- The series converges at x if

$$\lim_{k\to\infty} \sum_{n=0}^{k} a_n(x - x_0)^n$$

  exists for x. It clearly converges at x_0, but may or may not for other values of x

- The series **converges absolutely** if the following converges:

$$\sum_{n=0}^{\infty} |a_n(x - x_0)^n|$$

Ratio Test

Ratio Test: For the power series:

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n$$

- The **ratio test** provides a means of showing **absolute convergence**. If $a_n \neq 0$, x fixed, and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n(x - x_0)^n} \right| = |x - x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x - x_0|L,$$

  then the power series **converges absolutely** at x, if $|x - x_0|L < 1$.
  If $|x - x_0|L > 1$, then the series **diverges**.
  The test is **inconclusive** with $|x - x_0|L = 1$.

Example

Example: For the power series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}n(x-2)^n$$

The **ratio test** gives:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+1)(x-2)^{n+1}}{(-1)^{n+1}n(x-2)^n}\right| = |x-2|\lim_{n\to\infty}\left|\frac{n+1}{n}\right| = |x-2|.$$

This **converges absolutely** for $|x-2|<1$.
It **diverges** for $|x-2|L \geq 1$.

Radius of Convergence

Radius of Convergence: For the power series:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

typically, there is a positive number ρ, called the **radius of convergence**, such that the series **converges absolutely** for $|x - x_0| < \rho$ and **diverges** for $|x - x_0| > \rho$

Generally, we are not concerned about convergence at the endpoints

Properties of Series

Consider the series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = f(x) \quad \text{and} \quad \sum_{n=0}^{\infty} b_n(x-x_0)^n = g(x)$$

converging for $|x-x_0| < \rho$ with $\rho > 0$

- Two series can be added or subtracted for $|x-x_0| < \rho$

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x-x_0)^n$$

- Products can be done formally for $|x-x_0| < \rho$:

$$f(x)g(x) = \left[\sum_{n=0}^{\infty} a_n(x-x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n(x-x_0)^n\right] = \sum_{n=0}^{\infty} c_n(x-x_0)^n,$$

  where $c_n = a_0b_n + a_1b_{n-1} + \ldots + a_nb_0$

- Quotients are more complex, but can be handled similarly

Properties of Series

Suppose $f(x)$ satisfies the series below converging for $|x - x_0| < \rho$ with $\rho > 0$

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

- The function f is continuous and has derivatives of all orders:

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2},$$

  converging for $|x - x_0| < \rho$
- The value of a_n is

$$a_n = \frac{f^{(n)}(x_0)}{n!},$$

  the coefficients for the Taylor series for f. $f(x)$ is called **analytic**.
- If

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

  then $a_n = b_n$ for all n. If $f(x) = 0$, then $a_n = 0$ for all n

Series Solution near an Ordinary Point

Series Solution near an Ordinary Point, x_0

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

where P, Q, and R are polynomials

Assume $y = \phi(x)$ is a solution with a Taylor series

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

with convergence for $|x - x_0| < \rho$

Initial conditions: It is easy to see that

$$y(x_0) = a_0 \qquad \text{and} \qquad y'(x_0) = a_1$$

Series Solution near an Ordinary Point

Theorem

*If x_0 is an **ordinary point** of the differential equation:*

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

that is, if $p = Q/P$ and $q = R/P$ are analytic at x_0, then the general solution of the DE is

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1 + a_1 y_2,$$

*where a_0 and a_1 are arbitrary, and y_1 and y_2 are two **power series solutions** that are **analytic** at x_0. The solutions y_1 and y_2 form a **fundamental set**. Further, the **radius of convergence** for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p and q.*

Airy's Equation

Airy's Equation arises in optics, quantum mechanics, electromagnetics, and radiative transfer:

$$y'' - xy = 0$$

Assume a **power series solution** of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

From before,

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n,$$

which is substituted into the **Airy's equation**

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}$$

Airy's Equation 2

Airy's Equation: The series can be written

$$2 \cdot 1 a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=1}^{\infty} a_{n-1} x^n,$$

so $a_2 = 0$

The **recurrence relation** satisfies

$$(n+2)(n+1) a_{n+2} = a_{n-1} \qquad \text{or} \qquad a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)},$$

so $a_2 = a_5 = a_8 = ... = a_{3n+2} = 0$ with $n = 0, 1, ...$

For the sequence, a_0, a_3, a_6, ... with $n = 1, 4, ...$

$$a_3 = \frac{a_0}{2 \cdot 3}, \quad a_6 = \frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}, \quad a_9 = \frac{a_6}{8 \cdot 9} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}$$

Airy's Equation 3

Airy's Equation: The general formula is

$$a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)(3n)}, \qquad n \geq 4$$

For the sequence, a_1, a_4, a_7, ... with $n = 2, 5, ...$

$$a_4 = \frac{a_1}{3 \cdot 4}, \quad a_7 = \frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}, \quad a_{10} = \frac{a_7}{9 \cdot 10} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}$$

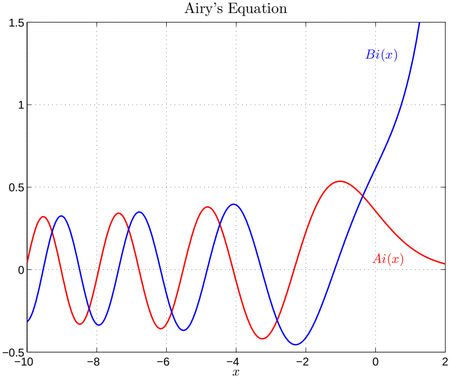
The general formula is

$$a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n)(3n+1)}, \qquad n \geq 4$$

Airy's Equation

Airy's Equation: The general solution is

$$y(x) = a_0\left[1+\frac{x^3}{2\cdot 3}+\frac{x^6}{2\cdot 3\cdot 5\cdot 6}+\cdots+\frac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdots(3n-1)(3n)}+\cdots\right]$$
$$+a_1\left[x+\frac{x^4}{3\cdot 4}+\frac{x^7}{3\cdot 4\cdot 6\cdot 7}+\cdots+\frac{x^{3n+1}}{3\cdot 4\cdot 6\cdot 7\cdots(3n)(3n+1)}+\cdots\right]$$



Chebyshev's Equation

Chebyshev's Equation is given by

$$(1-x^2)y'' - xy' + \alpha^2 y = 0$$

Let $\alpha = 4$ and try a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{so} \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

These are inserted into the **Chebyshev Equation** to give:

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + 16\sum_{n=0}^{\infty} a_n x^n = 0$$

Note that the first two sums could start their index at $n = 0$ without changing anything

Chebyshev's Equation 2

Chebyshev's Equation: The previous expression is easily changed by multiplying by x or x^2 and shifting the index to:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^n + 16\sum_{n=0}^{\infty} a_n x^n = 0$$

Equivalently,

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n(n-1)+n-16)\,a_n\right]x^n = 0$$

or

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - \left(n^2-16\right)a_n\right]x^n = 0$$

Chebyshev's Equation

Chebyshev's Equation: The previous expression gives the **recurrence relation**:

$$a_{n+2} = \frac{n^2 - 16}{(n+2)(n+1)} a_n \qquad \text{for} \quad n = 0, 1, ..$$

As before, a_0 and a_1 are arbitrary with $y(0) = a_0$ and $y'(0) = a_1$

It follows that

$$a_2 = -\frac{16}{2} a_0 = -8a_0, \qquad a_4 = \frac{4-16}{4 \cdot 3} a_2 = 8a_0, \qquad a_6 = 0 = a_8 = ... = a_{2n}$$

and

$$a_3 = -\frac{15}{3 \cdot 2} a_1 = -\frac{5}{2} a_1, \qquad a_5 = -\frac{7}{5 \cdot 4} a_3 = \frac{7}{8} a_1, \qquad a_7 = \frac{9}{7 \cdot 6} a_5 = \frac{3}{16} a_1, ...$$

Chebyshev's Equation

Chebyshev's Equation with $\alpha = 4$: From the **recurrence relation**, we see that the even series terminates after x^4, leaving a 4^{th} order polynomial solution.
The general solution becomes:

$$\begin{aligned} y(x) &= a_0\left(1 - 8x^2 + 8x^4\right) \\ &+ a_1\left(x - \frac{5}{2}x^3 + \frac{7}{8}x^5 + \frac{3}{16}x^7 + \dots\right) \end{aligned}$$

$$\begin{aligned} y(x) &= a_0\left(1 - 8x^2 + 8x^4\right) \\ &+ a_1\left(x + \sum_{n=1}^{\infty} \frac{[(2n-1)^2 - 16][(2n-3)^2 - 16] \cdot \dots \cdot (3^2 - 16)(1 - 16)}{(2n+1)!} x^{2n+1}\right) \end{aligned}$$

More generally, it is not hard to see that for any α an integer, the **Chebyshev's Equation** results in one solution being a polynomial of order α (only odd or even terms). The other solution is an infinite series.

The polynomial solution converges for all x, while the infinite series solution converges for $|x| < 1$.