Math 337 - Elementary Differential Equations

Lecture Notes – Numerical Methods for Differential Equations

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Outline

Introduction

Introduction

- Most differential equations can **not** be solved exactly
- Use the definition of the derivative to create a **difference equation**
- Develop numerical methods to solve differential equations
  - Euler's Method
  - Improved Euler's Method

Euler's Method

Initial Value Problem: Consider

$$\frac{dy}{dt} = f(t, y) \quad \text{with} \quad y(t_0) = y_0$$

- From the definition of the derivative

$$\frac{dy}{dt} = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}$$

- Instead of taking the limit, fix h, so

$$\frac{dy}{dt} \approx \frac{y(t+h) - y(t)}{h}$$

- Substitute into the differential equation and with algebra write

$$y(t+h) \approx y(t) + hf(t, y)$$

Euler's Method

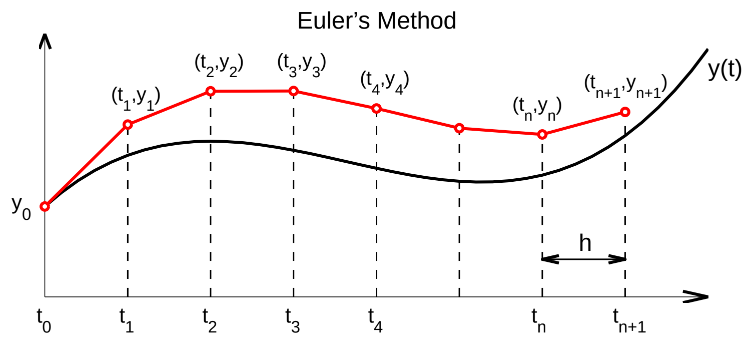
Euler's Method for a fixed h is

$$y(t+h) = y(t) + hf(t,y)$$

- Geometrically, Euler's method looks at the slope of the tangent line
  - The approximate solution follows the tangent line for a time step h
  - Repeat this process at each time step to obtain an approximation to the solution
- The ability of this method to track the solution accurately depends on the length of the time step, h, and the nature of the function $f(t,y)$
- This technique is rarely used as it has very bad convergence properties to the actual solution

Euler's Method

Graph of Euler's Method



Euler's Method

Euler's Method Formula: Euler's method is just a discrete dynamical system for approximating the solution of a continuous model

- Let $t_{n+1} = t_n + h$
- Define $y_n = y(t_n)$
- The initial condition gives $y(t_0) = y_0$
- **Euler's Method** is the discrete dynamical system

$$y_{n+1} = y_n + h\,f(t_n, y_n)$$

- Euler's Method only needs the initial condition to start and the right hand side of the differential equation (the **slope field**), $f(t,y)$ to obtain the approximate solution

Malthusian Growth Example

Malthusian Growth Example: Consider the model

$$\frac{dP}{dt} = 0.2\,P \qquad \text{with} \qquad P(0) = 50$$

Find the exact solution and approximate the solution with Euler's Method for $t \in [0,1]$ with $h = 0.1$

Solution: The exact solution is

$$P(t) = 50\,e^{0.2t}$$

Malthusian Growth Example

Solution (cont): The **Formula for Euler's Method** is

$$P_{n+1} = P_n + h\,0.2\,P_n$$

The initial condition $P(0) = 50$ implies that $t_0 = 0$ and $P_0 = 50$

Create a table for the Euler iterates

t_n	P_n
$t_0 = 0$	$P_0 = 50$
$t_1 = t_0 + h = 0.1$	$P_1 = P_0 + 0.1(0.2P_0) = 50 + 1 = 51$
$t_2 = t_1 + h = 0.2$	$P_2 = P_1 + 0.1(0.2P_1) = 51 + 1.02 = 52.02$
$t_3 = t_2 + h = 0.3$	$P_3 = P_2 + 0.1(0.2P_2) = 52.02 + 1.0404 = 53.0604$

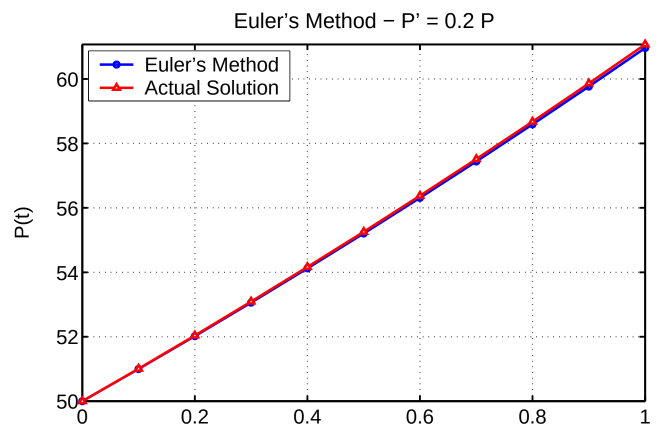
Malthusian Growth Example

Solution (cont): Iterations are easily continued - Below is table of the actual solution and the Euler's method iterates

t	Euler Solution	Actual Solution
0	50	50
0.1	51	51.01
0.2	52.02	52.041
0.3	53.060	53.092
0.4	54.122	54.164
0.5	55.204	55.259
0.6	56.308	56.375
0.7	57.434	57.514
0.8	58.583	58.676
0.9	59.755	59.861
1.0	60.950	61.070

Malthusian Growth Example

Graph of Euler's Method for Malthusian Growth Example



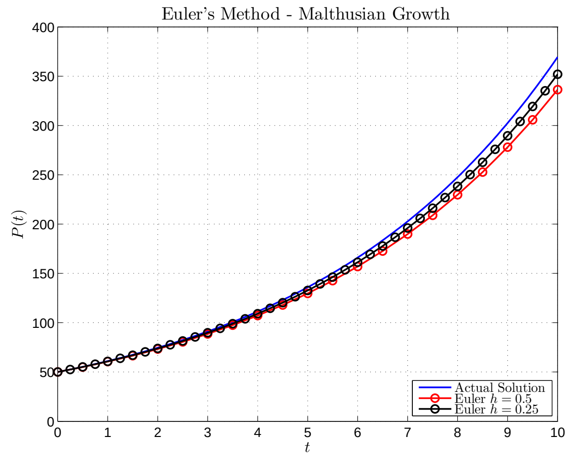
Malthusian Growth Example

Error Analysis and Larger Stepsize

- The table and the graph shows that Euler's method is tracking the solution fairly well over the interval of the simulation
- The error at $t = 1$ is only -0.2%
- However, this is a fairly short period of time and the stepsize is relatively small
- What happens when the stepsize is increased and the interval of time being considered is larger?

Malthusian Growth Example

Graph of Euler's Method with $h = 0.5$ and $h = 0.25$



There is a -9% error in the numerical solution at $t = 10$ for $h = 0.5$, and a -4.7% error when $h = 0.25$

Euler's Method - Algorithm

Algorithm (Euler's Method)

Consider the initial value problem

$$\frac{dy}{dt} = f(t,y), \qquad y(t_0) = y_0.$$

Let h be a fixed stepsize and define $t_n = t_0 + nh$. Also, let $y(t_n) = y_n$. ***Euler's Method*** *for approximating the solution to the IVP satisfies the* **difference equation**

$$y_{n+1} = y_n + hf(t_n, y_n).$$

Euler's Method - MatLab

Define a MatLab function for Euler's method for any function (func) with stepsize h, $t \in [t_0, t_f]$, and $y(t_0) = y_0$

```
function [t,y] = euler(func,h,t0,tf,y0)
% Euler's Method - Stepsize h, time from t0 to tf, initial y is y0

% Create time interval and initialize y
t = [t0:h:tf];
y(1) = y0;

% Loop for Euler's method
for i = 1:length(t)-1
    y(i+1) = y(i) + h*(feval(func,t(i),y(i)));
end

% Create column vectors t and y
t = t';
y = y';

end
```

Euler's Method - Population

Our initial example was $\frac{dP}{dt} = 0.2P$ with $P(0) = 50$

```
function z = pop(t,y)
% Malthusian growth
z = 0.2*y;
end
```

Create graph shown above

```
tt = linspace(0,10,200);
yy = 50*exp(0.2*tt);                   % Actual solution
[t,y]=euler(@pop,0.5,0,10,50);         % Implement Euler's method, 0.5
[t1,y1]=euler(@pop,0.25,0,10,50);  % Implement Euler's method, 0.25
plot(tt,yy,'b-','LineWidth',1.5);   % Actual solution
hold on                                  % Plots Multiple graphs
plot(t,y,'r-o','LineWidth',1.5,'MarkerSize',7);  % Euler h = 0.5
plot(t1,y1,'k-o','LineWidth',1.5,'MarkerSize',7); % Euler h = 0.25
grid                                       % Adds Gridlines
h = legend('Actual Solution','Euler $h = 0.5$','Euler $h = 0.25$',4);
set(h,'Interpreter','latex')  % Allow LaTeX in legend
axis([0 10 0 400]);  % Defines limits of graph
```

Euler's Method with $f(t,y)$ 1

Euler's Method with $f(t,y)$: Consider the model

$$\frac{dy}{dt} = y + t \qquad \text{with} \qquad y(0) = 3$$

Find the solution to this initial value problem

Rewrite this linear DE and find the integrating factor:

$$\frac{dy}{dt} - y = t \qquad \text{with} \qquad \mu(t) = e^{-t}$$

Solving

$$\frac{d}{dt}\left(e^{-t}y\right) = te^{-t} \qquad \text{or} \qquad e^{-t}y(t) = \int te^{-t}dt = -(t+1)e^{-t} + C$$

With the initial condition the solution is

$$y(t) = 4\,e^t - t - 1$$

Euler's Method with $f(t,y)$ 2

Solution (cont): **Euler's formula** with $h = 0.25$ is

$$y_{n+1} = y_n + 0.25(y_n + t_n)$$

t_n	Euler solution y_n
$t_0 = 0$	$y_0 = 3$
$t_1 = 0.25$	$y_1 = y_0 + h(y_0 + t_0) = 3 + 0.25(3 + 0) = 3.75$
$t_2 = 0.5$	$y_2 = y_1 + h(y_1 + t_1) = 3.75 + 0.25(3.75 + 0.25) = 4.75$
$t_3 = 0.75$	$y_3 = y_2 + h(y_2 + t_2) = 4.75 + 0.25(4.75 + 0.5) = 6.0624$
$t_4 = 1$	$y_4 = y_3 + h(y_3 + t_3) = 6.0624 + 0.25(6.0624 + 0.75) = 7.7656$

Actual solution is $y(1) = 8.8731$, so the Euler solution has a -12.5% error

If $h = 0.1$, after 10 steps $y(1) \approx y_{10} = 8.3750$ with -5.6% error

Euler's Method with $f(t,y)$ 3

Solution (cont): **Euler's formula** with different h is

$$y_{n+1} = y_n + h(y_n + t_n)$$

t_n	$h = 0.2$	$h = 0.1$	$h = 0.05$	$h = 0.025$	Actual
0.2	3.6	3.64	3.662	3.6736	3.6856
0.4	4.36	4.4564	4.5098	4.538	4.5673
0.6	5.312	5.4862	5.5834	5.6349	5.6885
0.8	6.4944	6.7744	6.9315	7.015	7.1022
1	7.9533	8.375	8.6132	8.7403	8.8731
2	21.7669	23.91	25.16	25.8383	26.5562
% Err	-18.0	-9.96	-5.26	-2.70	

We see the percent error at $t = 2$ (compared to the actual solution) declining by about $\frac{1}{2}$ as h is halved

Euler Error Analysis

- Consider the solution of the IVP $y' = f(t,y), \quad y(t_0) = y_0$ denoted $\phi(t)$
  - **Euler's formula**, $y_{n+1} = y_n + hf(t_n, y_n)$, approximates $y_n \approx \phi(t_n)$
  - Expect the **error** to decrease as h decreases
  - How small does h have to be to reach a certain tolerance?
- Errors
  - **Local truncation error**, e_n, is the amount of error at each step
  - **Global truncation error**, E_n, is the amount of error between the algorithm and $\phi(t)$
  - **Round-off error**, R_n, is the error due to the fact that computers hold finite digits

Local Truncation Error 1

Assume that $\phi(t)$ solves the IVP, so

$$\phi'(t) = f(t, \phi(t))$$

Use Taylor's theorem with a remainder, then

$$\phi(t_n + h) = \phi(t_n) + \phi'(t_n)h + \tfrac{1}{2}\phi''(\bar{t}_n)h^2,$$

where $\bar{t}_n \in (t_n, t_n + h)$

From ϕ being a solution of the IVP

$$\phi(t_{n+1}) = \phi(t_n) + hf(t_n, \phi(t_n)) + \tfrac{1}{2}\phi''(\bar{t}_n)h^2,$$

If $y_n = \phi(t_n)$ is the correct solution, then the **Euler approximate solution** at t_{n+1} is

$$y^*_{n+1} = \phi(t_n) + hf(t_n, \phi(t_n)),$$

so the **local truncation error** satisfies

$$e_{n+1} = \phi(t_{n+1}) - y^*_{n+1} = \tfrac{1}{2}\phi''(\bar{t}_n)h^2$$

Local Truncation Error 2

Since the **local truncation error** satisfies

$$e_{n+1} = \tfrac{1}{2}\phi''(\bar{t}_n)h^2,$$

then if there is a **uniform bound** $M = \max_{t\in[a,b]} |\phi''(t)|$, the local error is bounded with

$$|e_n| \le \frac{Mh^2}{2}$$

Thus, **Euler's Method** is said to have a **local truncation error of order** h^2 often denoted $\mathcal{O}(h^2)$

This result allows the choice of a stepsize to keep the numerical solution within a certain tolerance, say ε, or

$$\frac{Mh^2}{2} \le \varepsilon \qquad \text{or} \qquad h \le \sqrt{2\varepsilon/M}$$

Often difficult to estimate either $|\phi''(t)|$ or M

Global Truncation

Other Errors

- The **local truncation error** satisfies $|e_n| \le Mh^2/2$
  - This error is most significant for **adaptive numerical routines** where code is created to maintain a certain tolerance
- **Global Truncation Error**
  - The more important error for the numerical routines is this error over the entire simulation
  - **Euler's method** can be shown to have a **global truncation error**,
    $$|E_n| \le Kh$$
  - Note error is one order less than **local error**, which scales proportionally with the stepsize or $|E_n| \le \mathcal{O}(h)$
  - HW problem using Taylor's series and Math induction to prove this result

Global Truncation and Round-Off Error

Other Errors - continued

- **Round-Off Error**, R_n
  - This error results from the finite digits in the computer
  - All numbers in a computer are truncated
  - This is beyond the scope of this course
- **Total Computed Error**
  - The total error combines the machine error and the error of the algorithm employed
  - It follows that
    $$|\phi(t_n) - Y_n| \le |E_n| + |R_n|$$
  - The machine error cannot be controlled, but choosing a **higher order method** allows improving the **global truncation error**

Numerical solutions of DEs

Numerical solutions of differential equations

- Euler's Method is simple and intuitive, but lacks accuracy
- Numerical methods are available through standard software
  - MatLab's ode23
  - Maple's dsolve with *numeric* option
- Many types of numerical methods - different accuracies and stability
  - Easiest are **single stepsize Runge-Kutta methods**
  - Software above uses **adaptive stepsize Runge-Kutta methods**
  - Many other techniques shown in Math 542
- **Improved Euler's method** (or **Heun formula**) is a simple extension of Euler's method - However, significantly better

Improved Euler's Method - Algorithm

Algorithm (Improved Euler's Method (or Heun Formula))

Consider the initial value problem

$$\frac{dy}{dt} = f(t,y), \qquad y(t_0) = y_0.$$

Let h be a fixed stepsize. Define $t_n = t_0 + nh$ and the approximate solution $y(t_n) = y_n$.

1. *Approximate y by* ***Euler's Method***

$$ye_n = y_n + hf(t_n, y_n)$$

2. ***Improved Euler's Method*** *is the* **difference formula**

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n,y_n) + f(t_n+h, ye_n)\right)$$

Improved Euler's Method

Improved Euler's Method Formula: This technique is an easy extension of Euler's Method

- The Improved Euler's method uses an average of the Euler's method and an Euler's method approximation to the function
- This technique requires two function evaluations, instead of one
- Simple two step algorithm for implementation
- Can show this converges as $\mathcal{O}(h^2)$, which is significantly better than Euler's method

Improved Euler's Method - MatLab

Define a MatLab function for the Improved Euler's method for any function (func) with stepsize h, $t \in [t_0, t_f]$, and $y(t_0) = y_0$

```matlab
function [t,y] = im_euler(func,h,t0,tf,y0)
% Improved Euler's Method - Stepsize h, time from t0 to tf,
    initial y is y0
% Create time interval and initialize y
t = [t0:h:tf];
y(1) = y0;
% Loop for Improved Euler's method
for i = 1:length(t)-1
    ye = y(i) + h*(feval(func,t(i),y(i)));  % Euler's step
    y(i+1) = y(i) + (h/2)*(feval(func,t(i),y(i)) + feval(
        func,t(i+1),ye));
end
% Create column vectors t and y
t = t';
y = y';
end
```

Example: Improved Euler's Method

Example: Improved Euler's Method: Consider the initial value problem:

$$\frac{dy}{dt} = y + t \qquad \text{with} \qquad y(0) = 3$$

- The solution to this differential equation is

$$y(t) = 4\,e^t - t - 1$$

- Numerically solve this using Euler's Method and Improved Euler's Method using $h = 0.1$
- Compare these numerical solutions

Example: Improved Euler's Method

Solution: Let $y_0 = 3$, the Euler's formula is

$$y_{n+1} = y_n + h(y_n + t_n) = y_n + 0.1(y_n + t_n)$$

The Improved Euler's formula is

$$ye_n = y_n + h(y_n + t_n) = y_n + 0.1(y_n + t_n)$$

with

$$\begin{aligned} y_{n+1} &= y_n + \tfrac{h}{2}\left((y_n + t_n) + (ye_n + t_n + h)\right) \\ y_{n+1} &= y_n + 0.05\left(y_n + ye_n + 2\,t_n + 0.1\right) \end{aligned}$$

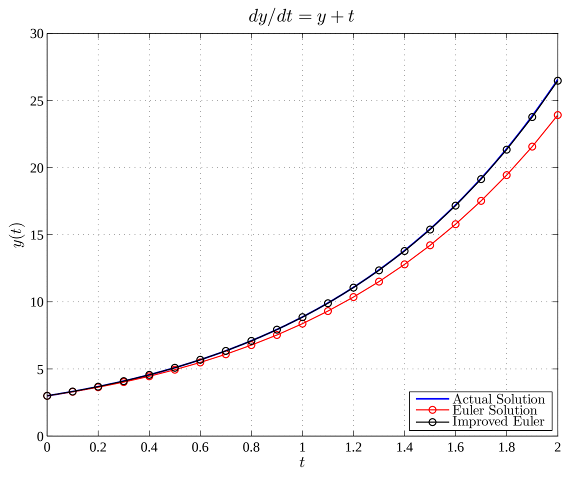
Example: Improved Euler's Method

Solution: Below is a table of the numerical computations

t	Euler's Method	Improved Euler	Actual
0	$y_0 = 3$	$y_0 = 3$	$y(0) = 3$
0.1	$y_1 = 3.3$	$y_1 = 3.32$	$y(0.1) = 3.3207$
0.2	$y_2 = 3.64$	$y_2 = 3.6841$	$y(0.2) = 3.6856$
0.3	$y_3 = 4.024$	$y_3 = 4.0969$	$y(0.3) = 4.0994$
0.4	$y_4 = 4.4564$	$y_4 = 4.5636$	$y(0.4) = 4.5673$
0.5	$y_5 = 4.9420$	$y_5 = 5.0898$	$y(0.5) = 5.0949$
0.6	$y_6 = 5.4862$	$y_6 = 5.6817$	$y(0.6) = 5.6885$
0.7	$y_7 = 6.0949$	$y_7 = 6.3463$	$y(0.7) = 6.3550$
0.8	$y_8 = 6.7744$	$y_8 = 7.0912$	$y(0.8) = 7.1022$
0.9	$y_9 = 7.5318$	$y_9 = 7.9247$	$y(0.9) = 7.9384$
1	$y_{10} = 8.3750$	$y_{10} = 8.8563$	$y(1) = 8.8731$

Example: Improved Euler's Method

Graph of Solution: Actual, Euler's and Improved Euler's



The Improved Euler's solution is very close to the actual solution

Example: Improved Euler's Method 5

Solution: Comparison of the numerical simulations

- It is very clear that the Improved Euler's method does a substantially better job of tracking the actual solution
- The Improved Euler's method requires only one additional function, $f(t, y)$, evaluation for this improved accuracy
- At $t = 1$, the Euler's method has a -5.6% error from the actual solution
- At $t = 1$, the Improved Euler's method has a -0.19% error from the actual solution

Improved Euler's Method Error

Improved Euler's Method Error

- Showed earlier that **Euler's method** had a **local truncation error** of $\mathcal{O}(h^2)$ with **global error** being $\mathcal{O}(h)$
- Similar **Taylor expansions** (in two variables) give the **local truncation error** for the **Improved Euler's method** as $\mathcal{O}(h^3)$
- For **Improved Euler's method**, the **global truncation error** is $\mathcal{O}(h^2)$
- From a practical perspective, these results imply:
  - With **Euler's method**, the reduction of the stepsize by a factor of 0.1 gains one digit of accuracy
  - With **Improved Euler's method**, the reduction of the stepsize by a factor of 0.1 gains two digits of accuracy
  - This is a **significant improvement** at only the cost of one additional function evaluation per step

Numerical Example 1

Numerical Example: Consider the IVP

$$\frac{dy}{dt} = 2e^{-0.1t} - \sin(y), \qquad y(0) = 3,$$

which has no exact solution, so must solve numerically

- Solve this problem with **Euler's method** and **Improved Euler's method**
- Show differences with different stepsizes for $t \in [0, 5]$
- Show the order of convergence by halving the stepsize twice
- Graph the solution and compare to solution from *ode23* in MatLab, closely approximating the exact solution

Numerical Example 2

Numerical Solution for $\frac{dy}{dt} = 2e^{-0.1t} - \sin(y), \quad y(0) = 3$

Used MatLab's *ode45* to obtain an accurate numerical solution to compare **Euler's method** and **Improved Euler's method** with stepsizes $h = 0.2$, $h = 0.1$, and $h = 0.05$

	"Actual"	Euler	Im Eul	Euler	Im Eul	Euler	Im Eul
t_n		$h = 0.2$	$h = 0.2$	$h = 0.1$	$h = 0.1$	$h = 0.05$	$h = 0.05$
0	3	3	3	3	3	3	3
1	5.5415	5.4455	5.5206	5.4981	5.5361	5.5209	5.5401
2	7.1032	7.1718	7.0881	7.1368	7.0995	7.1199	7.1023
3	7.753	7.836	7.743	7.7939	7.7505	7.7734	7.7524
4	8.1774	8.2818	8.167	8.2288	8.1748	8.2029	8.1768
5	8.5941	8.7558	8.5774	8.6737	8.5899	8.6336	8.5931
		1.88%	−0.194%	0.926%	−0.0489%	0.460%	−0.0116%

Last row shows percent error between the different approximations and the accurate solution

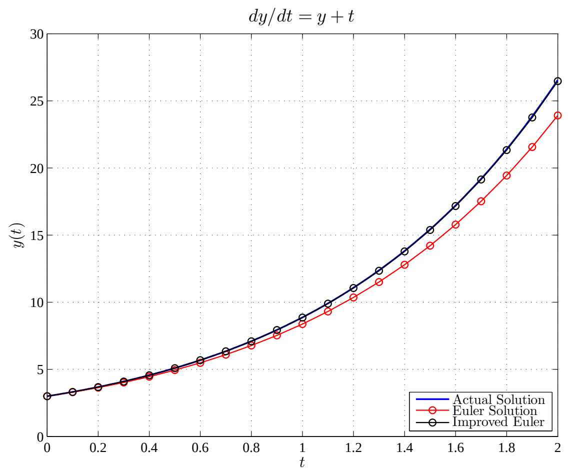
Numerical Example

Error of Numerical Solutions

- Observe that the **Improved Euler's method** with stepsize $h = 0.2$ is more accurate at $t = 5$ than **Euler's method** with stepsize $h = 0.05$
- With **Euler's method** the error cuts in half with halving of the stepsize
- With the **Improved Euler's method** the errors cuts in quarter with halving of the stepsize

Numerical Example

Graph of Solution: Actual, Euler's and Improved Euler's methods with $h = 0.2$



The Improved Euler's solution is very close to the actual solution

Order of Error

Error of Numerical Solutions

- **Order of Error** without good "Actual solution"
  - Simulate system with stepsizes h, $h/2$, and $h/4$ and define these simulates as y_n^1, y_n^2, and y_n^3, respectively
  - Compute the ratio (from Cauchy sequence)

$$R = \frac{|y_n^3 - y_n^2|}{|y_n^2 - y_n^1|}$$

  - If the numerical method is **order** m, then this ratio is approximately $\frac{1}{2^m}$
  - Above example at $t = 5$ has $R = 0.488$ for **Euler's method** and $R = 0.256$ for **Improved Euler's method**
  - Allows user to determine how much error numerical routine is generating