

Math 337 - Elementary Differential Equations

Lecture Notes – Systems of Two First Order Equations: Part B

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Outline

- 1 Introduction
- 2 Solutions of Two 1st Order Linear DEs
 - Existence and Uniqueness
- 3 Homogeneous Linear System of Autonomous DEs
 - Superposition and Linear Independence
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 - Real and Different Eigenvalues
 - Complex Eigenvalues
 - Repeated Eigenvalues
 - Bifurcation Example and Stability Diagram

Introduction

Introduction

- This is the second part of notes for **Systems of Two 1st Order Differential Equations**
- Part A has the topics below
 - A motivating example of a Greenhouse/Rockbed system of passive heating
 - Solutions for the example above - illustrating key techniques
 - Graphs for **direction fields** and **phase portraits**
 - **MatLab** and **Maple** introduced for these problems
- Part B has the following topics
 - Definitions and theorems for **Systems of Two 1st Order Differential Equations**
 - Superposition and linear independence
 - Solving with **eigenvalue techniques**
 - Analysis of different cases with their phase portraits

General Linear System - 2D

General System of Two 1st Order Linear DEs

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} p_{11}(t)x_1 + p_{12}(t)x_2 + g_1(t) \\ p_{21}(t)x_1 + p_{22}(t)x_2 + g_2(t) \end{pmatrix}, \quad (1)$$

which can be written

$$\dot{\mathbf{x}} = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t),$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix}, \quad \text{and} \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

System (1) is a 1st order linear system of DEs of dimension 2

If $\mathbf{g}(t) = \mathbf{0}$, then System (1) is **homogeneous**; otherwise it is **nonhomogeneous**

Two 1st Order Linear DEsExistence and Uniqueness for Two 1st Order Linear DEs

Theorem (Existence and Uniqueness)

Let each of the functions p_{11}, \dots, p_{22} , g_1 , and g_2 be continuous on an open interval $I = \{t | t \in (\alpha, \beta)\}$, let t_0 be any point in I , and let x_{10} and x_{20} be any given numbers. Then there exists a unique solution to the system (1):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} p_{11}(t)x_1 + p_{12}(t)x_2 + g_1(t) \\ p_{21}(t)x_1 + p_{22}(t)x_2 + g_2(t) \end{pmatrix},$$

that also satisfies the initial conditions

$$x_1(t_0) = x_{10}, \quad x_2(t_0) = x_{20}.$$

Further the solution exists throughout the interval I .

Linear Autonomous System

Linear Autonomous System: If the coefficient matrix \mathbf{P} and vector function \mathbf{g} are independent of time, *i.e.*, **constants**, then we have the **linear autonomous system**

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b},$$

with constant matrix \mathbf{A} and constant vector \mathbf{b} .

The **equilibrium solutions** or **critical points** are found by solving:

$$\mathbf{A}\mathbf{x}_e = -\mathbf{b} \quad \text{or} \quad \mathbf{x}_e = -\mathbf{A}^{-1}\mathbf{b}.$$

The change of variables $\mathbf{y} = \mathbf{x} - \mathbf{x}_e$ allows us to concentrate on the **homogeneous linear system with constant coefficients**

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$$

Superposition Principle

Theorem (Superposition Principle)

Suppose that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions of the equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t).$$

Then the expression

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t),$$

where c_1 and c_2 are arbitrary constants, is also a solution.

We use the linearity of differentiation and matrices to show this

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \frac{d}{dt} (c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)) = c_1\dot{\mathbf{x}}_1(t) + c_2\dot{\mathbf{x}}_2(t) \\ &= c_1\mathbf{A}\mathbf{x}_1(t) + c_2\mathbf{A}\mathbf{x}_2(t) = \mathbf{A} (c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)) = \mathbf{A}\mathbf{x}(t)\end{aligned}$$

Wronskian and Linear Independence

Definition (Wronskian)

Suppose that $\mathbf{x}_1(t) = [x_{11}(t), x_{21}(t)]^T$ and $\mathbf{x}_2(t) = [x_{12}(t), x_{22}(t)]^T$. The **Wronskian** of the solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ is given by the determinant

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}$$

Definition (Linear Independence of Solutions)

Suppose that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions of $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ on some interval I . We say that \mathbf{x}_1 and \mathbf{x}_2 are **linearly dependent** if there exists a constant k such that

$$\mathbf{x}_1(t) = k\mathbf{x}_2(t), \quad \text{for all } t \text{ in } I.$$

Otherwise, \mathbf{x}_1 and \mathbf{x}_2 are **linearly independent**.

Wronskian and Linear Independence

Theorem (Wronskian and Linear Independence)

Suppose that

$$\mathbf{x}_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix}$$

are solutions of $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ on an interval I . Then \mathbf{x}_1 and \mathbf{x}_2 are **linearly independent** if and only if the **Wronskian**

$$W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0, \quad \text{for all } t \text{ in } I.$$

The two linearly independent solutions of $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ are often called a **fundamental set of solutions**

Fundamental Solutions

Theorem (Fundamental Solutions)

Suppose that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two solutions of

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \quad (2)$$

and that their Wronskian is not zero on an interval I . Then \mathbf{x}_1 and \mathbf{x}_2 form a **fundamental set of solutions** for (2), and the general solution is given by

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t),$$

where c_1 and c_2 are arbitrary constants. If there is a given initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, where \mathbf{x}_0 is any constant vector, then this condition determines the constants c_1 and c_2 uniquely.

Solving $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

Consider the general problem

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t),$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

We attempt a solution of the form

$$\mathbf{x} = e^{\lambda t} \mathbf{v}, \quad \text{so} \quad \lambda e^{\lambda t} \mathbf{v} = \mathbf{A} e^{\lambda t} \mathbf{v}$$

Since $e^{\lambda t}$ is never zero,

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \quad \text{or} \quad (\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0},$$

where \mathbf{I} is the 2×2 identity matrix

This is the classic **eigenvalue problem**

Eigenvalue Problem

Thus, solving the homogeneous DE $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is equivalent to solving the **eigenvalue problem**

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0} \quad \text{with} \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

From Linear Algebra (Math 254) the **eigenvalues** are found by solving

$$\det |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0.$$

This gives the **characteristic equation**

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

This is a quadratic equation, so easily solved for λ_1 and λ_2

Each λ_i is inserted into $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, and the corresponding **eigenvectors**, \mathbf{v}_i are found

Real and Different Eigenvalues

Consider $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ and assume that the **eigenvalue problem** $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ has **real and different eigenvalues**, λ_1 and λ_2

The two solutions are

$$\mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1 \quad \text{and} \quad \mathbf{x}_2(t) = e^{\lambda_2 t} \mathbf{v}_2,$$

so the **Wronskian** is

$$W[\mathbf{x}_1(t), \mathbf{x}_2(t)](t) = \begin{vmatrix} v_{11}e^{\lambda_1 t} & v_{12}e^{\lambda_2 t} \\ v_{21}e^{\lambda_1 t} & v_{22}e^{\lambda_2 t} \end{vmatrix} = \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix} e^{(\lambda_1 + \lambda_2)t}$$

Since $e^{(\lambda_1 + \lambda_2)t}$ is nonzero, the Wronskian is nonzero if and only if $\det[\mathbf{v}_1, \mathbf{v}_2] \neq 0$.

Recall if the Wronskian is nonzero, then $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ form a **fundamental set of solutions** to the system of DEs

Linear Algebra Result

Theorem

Let \mathbf{A} have real or complex eigenvalues, λ_1 and λ_2 , such that $\lambda_1 \neq \lambda_2$, and let the corresponding eigenvectors be

$$\mathbf{v}_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}.$$

If \mathbf{V} is the matrix formed from \mathbf{v}_1 and \mathbf{v}_2 with

$$\mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix},$$

then

$$\det |\mathbf{V}| = \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix} \neq 0.$$

Real and Different Eigenvalues

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The two previous slides show that if \mathbf{A} has **real and different eigenvalues**, λ_1 and λ_2 , then the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

has a **fundamental set of solutions**

$$\mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1 \quad \text{and} \quad \mathbf{x}_2(t) = e^{\lambda_2 t} \mathbf{v}_2,$$

where \mathbf{v}_1 and \mathbf{v}_2 are the corresponding eigenvectors for λ_1 and λ_2 , respectively

It follows that the general solution can be written

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2.$$

Real and Different Eigenvalues

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Example 1: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -0.5 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

From above we need to find the eigenvalues and eigenvectors, so solve

$$\det \begin{vmatrix} -0.5 - \lambda & 2 \\ 0 & -1 - \lambda \end{vmatrix} = (\lambda + 0.5)(\lambda + 1) = 0,$$

which is the **characteristic equation** with solutions $\lambda_1 = -0.5$ and $\lambda_2 = -1$

Real and Different Eigenvalues

Example 1 (cont): For $\lambda_1 = -0.5$ we have:

$$\begin{pmatrix} -0.5 - \lambda_1 & 2 \\ 0 & -1 - \lambda_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & -0.5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Similarly, for $\lambda_2 = -1$ we have:

$$\begin{pmatrix} -0.5 - \lambda_2 & 2 \\ 0 & -1 - \lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0.5 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(2)} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

Real and Different Eigenvalues

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Example 1 (cont): The results above give the general solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-0.5t} + c_2 \begin{pmatrix} 4 \\ -1 \end{pmatrix} e^{-t},$$

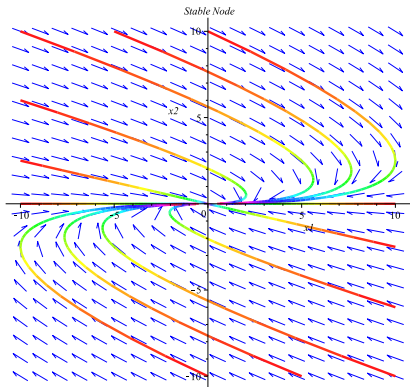
which is a solution exponentially decaying toward the origin.

This is a **sink** or **stable node**.

Solutions move rapidly

in the direction $\xi^{(2)} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$,

while decaying more slowly in the direction $\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



Real and Different Eigenvalues

Example 2: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

From above we need to find the eigenvalues and eigenvectors, so solve

$$\det \begin{vmatrix} -\lambda & 1 \\ -3 & 4 - \lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0,$$

which is the **characteristic equation** with solutions $\lambda_1 = 1$ and $\lambda_2 = 3$

Real and Different Eigenvalues

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Example 2 (cont): For $\lambda_1 = 1$ we have:

$$\begin{pmatrix} -\lambda_1 & 1 \\ -3 & 4 - \lambda_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Similarly, for $\lambda_2 = 3$ we have:

$$\begin{pmatrix} -\lambda_2 & 1 \\ -3 & 4 - \lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Real and Different Eigenvalues

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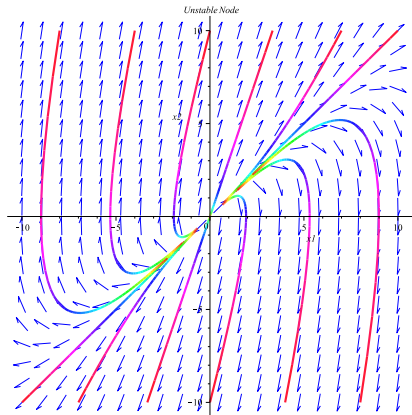
Example 2 (cont): The results above give the general solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t},$$

which is a solution exponentially growing away from the origin.

This is a **source** or **unstable node**.

Solutions first move away from the origin in the direction $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then asymptotically parallel the direction $\xi^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ for larger t



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Real and Different Eigenvalues

Example 3: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

From above we need to find the eigenvalues and eigenvectors, so solve

$$\det \begin{vmatrix} 1 - \lambda & 3 \\ 1 & -1 - \lambda \end{vmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2) = 0,$$

which is the **characteristic equation** with solutions $\lambda_1 = 2$ and $\lambda_2 = -2$

Real and Different Eigenvalues

Example 3 (cont): For $\lambda_1 = 2$ we have:

$$\begin{pmatrix} 1 - \lambda_1 & 3 \\ 1 & -1 - \lambda_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(1)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Similarly, for $\lambda_2 = -2$ we have:

$$\begin{pmatrix} 1 - \lambda_2 & 3 \\ 1 & -1 - \lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Real and Different Eigenvalues

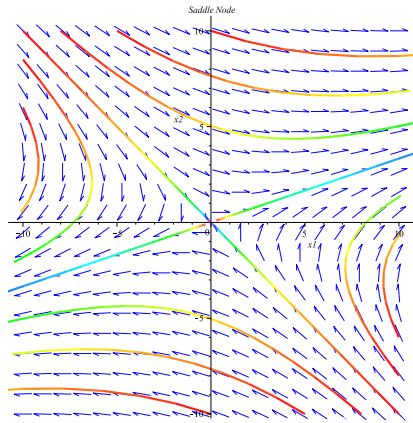
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Example 3 (cont): The results above give the general solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}.$$

This is a **saddle node**.

Solutions move toward the origin
 in the direction $\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
 and move away from origin in the
 direction $\xi^{(1)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for larger t



Real and Different Eigenvalues

Example 4: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

If we seek equilibria, then

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_{1e} \\ x_{2e} \end{pmatrix}$$

However, any solution of the form $x_{1e} = 2x_{2e}$ is a **critical point**, giving a line of **equilibria**

Our method from before still applies, so seek $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$, which gives the **eigenvalue problem** below

$$\det \begin{vmatrix} -2 - \lambda & 4 \\ 1 & -2 - \lambda \end{vmatrix} = \lambda^2 + 4\lambda = \lambda(\lambda + 4) = 0,$$

has the **characteristic equation** with eigenvalues $\lambda = 0, -4$

Real and Different Eigenvalues

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Example 4 (cont): For $\lambda_1 = 0$ we have:

$$\begin{pmatrix} -2 - \lambda_1 & 4 \\ 1 & -2 - \lambda_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Similarly, for $\lambda_2 = -4$ we have:

$$\begin{pmatrix} -2 - \lambda_2 & 4 \\ 1 & -2 - \lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(2)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Real and Different Eigenvalues

Example 4 (cont): The **eigenvalue problem** gives two solutions to the DE

$$\mathbf{x}_1(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-4t}$$

The **Wronskian** satisfies

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{vmatrix} 2 & 2e^{-4t} \\ 1 & -e^{-4t} \end{vmatrix} = -4e^{-4t} \neq 0,$$

so these do form a **fundamental set of solutions**

Thus the general solution is given by

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-4t}.$$

Real and Different Eigenvalues

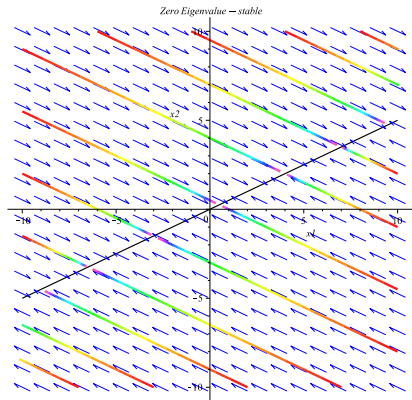
Example 4 (cont): The phase portrait for

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-4t}.$$

This is a **degenerate** case where the line $x_1 = 2x_2$ all form **equilibria**.

All solutions **exponentially approach** one of the equilibria along lines parallel to the line $x_1 = -2x_2$

Note: There is an **unstable case**, which we omit, where the eigenvalues satisfy $\lambda_1 = 0$ and $\lambda_2 > 0$



Complex Eigenvalues

1

Consider a system of two linear homogeneous differential equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

where \mathbf{A} is a real-valued matrix.

With a solution of the form $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$, there are **eigenvalues**, λ , with corresponding **eigenvectors**, \mathbf{v} satisfying

$$\det |\mathbf{A} - \lambda\mathbf{I}| = 0 \quad \text{and} \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

The **characteristic equation** for the **eigenvalues** is a quadratic equation.

Assume the eigenvalues are complex, then $\lambda = \mu \pm i\nu$, since \mathbf{A} is real-valued

Complex Eigenvalues

2

Assume the DE, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, has eigenvalues $\lambda_1 = \mu + i\nu$ and $\lambda_2 = \bar{\lambda}_1 = \mu - i\nu$

Assume \mathbf{v}_1 is an eigenvector corresponding to λ_1 , so

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_1 = \mathbf{0}$$

Taking **conjugates** (with \mathbf{A} , \mathbf{I} , and $\mathbf{0}$, real)

$$(\mathbf{A} - \bar{\lambda}_1 \mathbf{I})\bar{\mathbf{v}}_1 = (\mathbf{A} - \lambda_2 \mathbf{I})\bar{\mathbf{v}}_1 = \mathbf{0}$$

This gives two complex solutions to the system of DEs

$$\mathbf{x}_1(t) = e^{(\mu+i\nu)t}\mathbf{v}_1 \quad \text{and} \quad \mathbf{x}_2(t) = e^{(\mu-i\nu)t}\bar{\mathbf{v}}_1$$

We use **Euler's formula** to separate the solutions into real and imaginary parts

$$e^{i\nu t} = \cos(\nu t) + i \sin(\nu t)$$

Complex Eigenvalues

3

Assume the **eigenvector**, $\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}$, where \mathbf{a} and \mathbf{b} are real-valued, then

$$\begin{aligned}\mathbf{x}_1(t) &= (\mathbf{a} + i\mathbf{b})e^{\mu t}(\cos(\nu t) + i \sin(\nu t)) \\ &= e^{\mu t}(\mathbf{a} \cos(\nu t) - \mathbf{b} \sin(\nu t)) + ie^{\mu t}(\mathbf{a} \sin(\nu t) + \mathbf{b} \cos(\nu t))\end{aligned}$$

Denote the real and imaginary parts of $\mathbf{x}_1(t) = \mathbf{u}(t) + i\mathbf{w}(t)$

$$\mathbf{u}(t) = e^{\mu t}(\mathbf{a} \cos(\nu t) - \mathbf{b} \sin(\nu t)) \quad \text{and} \quad \mathbf{w}(t) = e^{\mu t}(\mathbf{a} \sin(\nu t) + \mathbf{b} \cos(\nu t))$$

A similar calculation gives

$$\mathbf{x}_2(t) = \mathbf{u}(t) - i\mathbf{w}(t),$$

so $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are complex conjugates.

The desire is to show that $\mathbf{u}(t)$ and $\mathbf{w}(t)$ are real-valued solutions forming a **fundamental set** for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

Complex Eigenvalues

Since $\mathbf{x}_1(t) = \mathbf{u}(t) + i\mathbf{w}(t)$ is a solution to the DE $\dot{\mathbf{x}}_1 = \mathbf{A}\mathbf{x}_1$, we have

$$\begin{aligned} \mathbf{0} &= \dot{\mathbf{x}}_1 - \mathbf{A}\mathbf{x}_1 = (\dot{\mathbf{u}} + i\dot{\mathbf{w}}) - \mathbf{A}(\mathbf{u} + i\mathbf{w}) \\ &= (\dot{\mathbf{u}} - \mathbf{A}\mathbf{u}) + i(\dot{\mathbf{w}} - \mathbf{A}\mathbf{w}) \end{aligned}$$

This vector is zero if and only if the real and imaginary parts are zero, so

$$\dot{\mathbf{u}} - \mathbf{A}\mathbf{u} = \mathbf{0} \quad \text{and} \quad \dot{\mathbf{w}} - \mathbf{A}\mathbf{w} = \mathbf{0}$$

or $\mathbf{u}(t)$ and $\mathbf{w}(t)$ are real-valued solutions of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

It remains to show $\mathbf{u}(t)$ and $\mathbf{w}(t)$ form a **fundamental set of solutions**, which is done with the **Wronskian**

Complex Eigenvalues

The two solutions are

$$\mathbf{u}(t) = e^{\mu t}(\mathbf{a} \cos(\nu t) - \mathbf{b} \sin(\nu t)) \quad \text{and} \quad \mathbf{w}(t) = e^{\mu t}(\mathbf{a} \sin(\nu t) + \mathbf{b} \cos(\nu t)),$$

so let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then the **Wronskian** satisfies

$$\begin{aligned} W[\mathbf{u}, \mathbf{w}](t) &= \begin{vmatrix} e^{\mu t}(a_1 \cos(\nu t) - b_1 \sin(\nu t)) & e^{\mu t}(a_1 \sin(\nu t) + b_1 \cos(\nu t)) \\ e^{\mu t}(a_2 \cos(\nu t) - b_2 \sin(\nu t)) & e^{\mu t}(a_2 \sin(\nu t) + b_2 \cos(\nu t)) \end{vmatrix} \\ &= (a_1 b_2 - a_2 b_1) e^{2\mu t} \end{aligned}$$

Assume $\nu \neq 0$ and the eigenvectors are $\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}$ and $\mathbf{v}_2 = \mathbf{a} - i\mathbf{b}$,

$$\begin{vmatrix} a_1 + ib_1 & a_1 - ib_1 \\ a_2 + ib_2 & a_2 - ib_2 \end{vmatrix} = -2i(a_1 b_2 - a_2 b_1) \neq 0$$

by our Theorem from Linear Algebra

Thus, the **Wronskian** shows $\mathbf{u}(t)$ and $\mathbf{w}(t)$ form a **fundamental set of solutions** to our problem

Complex Eigenvalues

Example 5: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

From above we need to find the eigenvalues and eigenvectors, so solve

$$\det \begin{vmatrix} 3 - \lambda & -2 \\ 4 & -1 - \lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5 = 0,$$

which is the **characteristic equation** with solutions $\lambda = 1 \pm 2i$
(complex eigenvalues)

Complex Eigenvalues

Example 5 (cont): For $\lambda_1 = 1 + 2i$ we have:

$$\begin{pmatrix} 3 - \lambda_1 & -2 \\ 4 & -1 - \lambda_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2 - 2i & -2 \\ 4 & -2 - 2i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}$.

We have $\lambda_2 = \bar{\lambda}_1$ and $\xi^{(2)} = \bar{\xi}^{(1)}$

Thus,

$$\begin{aligned} \mathbf{x}_1(t) &= \begin{pmatrix} 1 \\ 1 - i \end{pmatrix} e^t (\cos(2t) + i \sin(2t)) = \\ \mathbf{u}(t) + i\mathbf{w}(t) &= \begin{pmatrix} e^t \cos(2t) \\ e^t (\cos(2t) + \sin(2t)) \end{pmatrix} + i \begin{pmatrix} e^t \sin(2t) \\ e^t (\sin(2t) - \cos(2t)) \end{pmatrix} \end{aligned}$$

Complex Eigenvalues

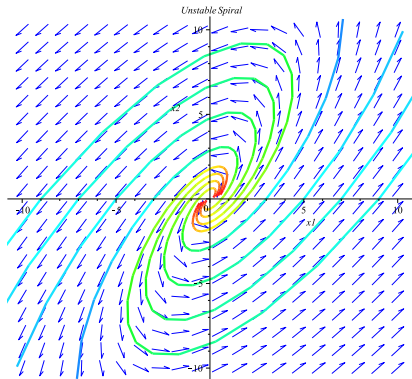
Example 5 (cont): From above the general solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} e^t \cos(2t) \\ e^t(\cos(2t) + \sin(2t)) \end{pmatrix} + c_2 \begin{pmatrix} e^t \sin(2t) \\ e^t(\sin(2t) - \cos(2t)) \end{pmatrix}.$$

This is an **unstable spiral**.

All solutions spiral away from the origin.

Solutions with complex eigenvalues with negative real parts spiral toward the origin, creating a **stable spiral**



Imaginary Eigenvalues

Example 6: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

From above we need to find the eigenvalues and eigenvectors, so solve

$$\det \begin{vmatrix} 2 - \lambda & -5 \\ 1 & -2 - \lambda \end{vmatrix} = \lambda^2 + 1 = 0,$$

which is the **characteristic equation** with solutions $\lambda = \pm i$ (purely imaginary eigenvalues)

Imaginary Eigenvalues

10

Example 6 (cont): For $\lambda_1 = i$ we have:

$$\begin{pmatrix} 2 - \lambda_1 & -5 \\ 1 & -2 - \lambda_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2 - i & -5 \\ 1 & -2 - i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$.

We have $\lambda_2 = \bar{\lambda}_1$ and $\xi^{(2)} = \bar{\xi}^{(1)}$

Thus,

$$\begin{aligned} \mathbf{x}_1(t) &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} (\cos(t) + i \sin(t)) = \\ \mathbf{u}(t) + i\mathbf{w}(t) &= \begin{pmatrix} 2 \cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} + i \begin{pmatrix} 2 \sin(t) + \cos(t) \\ \sin(t) \end{pmatrix} \end{aligned}$$

Imaginary Eigenvalues

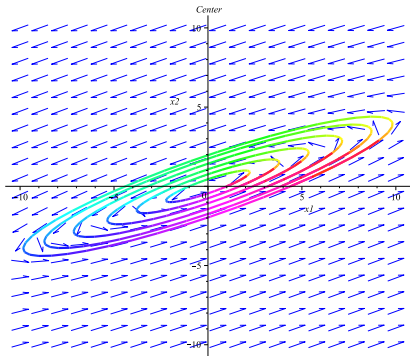
11

Example 6 (cont): From above the general solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} + c_2 \begin{pmatrix} 2 \sin(t) + \cos(t) \\ \sin(t) \end{pmatrix}.$$

This is a **center**.

All solutions form ellipses
around the origin.



Repeated Eigenvalues

1

Example 7: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

From above we need to find the eigenvalues and eigenvectors, so solve

$$\det \begin{vmatrix} 2 - \lambda & 0 \\ 0 & 2 - \lambda \end{vmatrix} = (\lambda - 2)^2 = 0,$$

which has the **characteristic equation** with solutions $\lambda = 2$ with an **algebraic multiplicity** of **2**

Repeated Eigenvalues

2

Example 7 (cont): For $\lambda_1 = \lambda_2 = 2$ we have:

$$\begin{pmatrix} 2 - \lambda_1 & 0 \\ 0 & 2 - \lambda_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus, $\lambda = 2$ has a **geometric multiplicity** of **2**, so the **eigenspace** for $\lambda = 2$ has dimension 2.

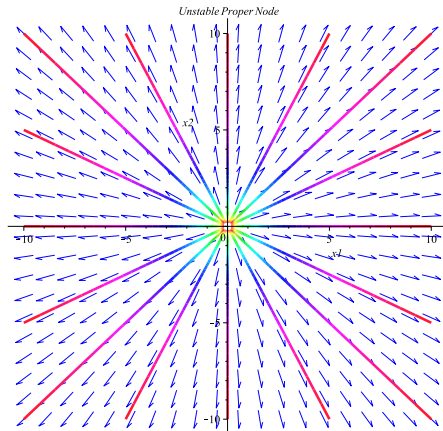
It follows that we can select the standard basis vectors as our eigenvectors, which gives the general solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}.$$

Repeated Eigenvalues

3

Example 7 (cont): This DE produces an **unstable proper node** or **star node** with all solutions following straight paths away from the origin



Repeated Eigenvalues

Example 8: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

This is an **upper triangular matrix**, so its eigenvalues are the diagonal elements.

Thus, $\lambda = -1$ with an **algebraic multiplicity** of **2**

$$\begin{pmatrix} -1 - \lambda & 1 \\ 0 & -1 - \lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This system only has the **1** eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Repeated Eigenvalues

Example 8 (cont): Since there is only one eigenvector, we obtain the one solution

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{-t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$$

Thus, $\lambda = -1$ has a **geometric multiplicity** of 1, so the **eigenspace** for $\lambda = -1$ has dimension 1.

If we examine the scalar equations, then

$$\dot{x}_1 = -x_1 + x_2 \quad \text{and} \quad \dot{x}_2 = -x_2$$

Thus, $x_2(t) = c_2 e^{-t}$, so

$$\dot{x}_1 + x_1 = c_2 e^{-t} \quad \text{with} \quad \mu(t) = e^t$$

This has the solution

$$x_1(t) = c_2 t e^{-t} + c_1 e^{-t}$$

Repeated Eigenvalues

Example 8 (cont): Combining the results above we see

$$\begin{aligned}\mathbf{x}(t) &= \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix} e^{-t} \\ &= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] e^{-t}\end{aligned}$$

The second solution has the form

$$\mathbf{x}_2(t) = \mathbf{v}te^{-t} + \mathbf{w}e^{-t}$$

Upon differentiation

$$\dot{\mathbf{x}}_2(t) = \mathbf{v}(1-t)e^{-t} - \mathbf{w}e^{-t} = \mathbf{A}\mathbf{x}_2 = \mathbf{A}(\mathbf{v}te^{-t} + \mathbf{w}e^{-t})$$

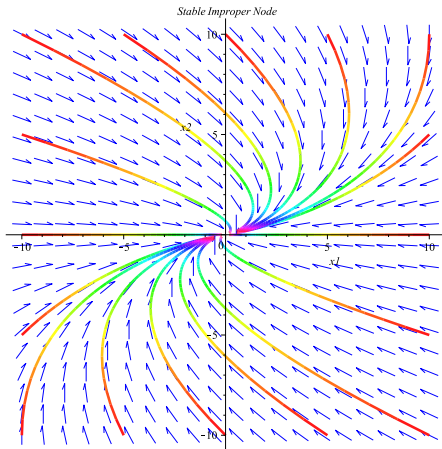
Since $(\mathbf{A} + \mathbf{I})\mathbf{v} = \mathbf{0}$, this reduces to solving for \mathbf{w}

$$(\mathbf{A} + \mathbf{I})\mathbf{w} = \mathbf{v} \quad \text{or} \quad \mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Repeated Eigenvalues

7

Example 8 (cont): This DE produces a **stable improper node** with all solutions moving toward the origin



Repeated Eigenvalues - General

Repeated Eigenvalues - Two Dimensional Null Space

Suppose the 2×2 matrix \mathbf{A} has a repeated eigenvalue λ .

If the eigenspace spanned by the eigenvectors has dimension 2, \mathbf{v}_1 and \mathbf{v}_2 , then the solution is simply

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda t} + c_2 \mathbf{v}_2 e^{\lambda t}$$

Repeated Eigenvalues - General

Repeated Eigenvalues - One Dimensional Null Space If the 2×2 matrix \mathbf{A} has only one eigenvector \mathbf{v} associated with λ , then one solution is

$$\mathbf{x}_1(t) = \mathbf{v}e^{\lambda t}$$

We attempt a second solution of the form

$$\mathbf{x}_2(t) = \mathbf{v}te^{\lambda t} + \mathbf{w}e^{\lambda t},$$

which upon differentiation gives

$$\dot{\mathbf{x}}_2(t) = \mathbf{v}(\lambda t + 1)e^{\lambda t} + \lambda \mathbf{w}e^{\lambda t} = \mathbf{A}\mathbf{x}_2 = \mathbf{A}(\mathbf{v}te^{\lambda t} + \mathbf{w}e^{\lambda t})$$

Since $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, this reduces to solving for \mathbf{w}

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}$$

This gives the second linearly independent solution, $\mathbf{x}_2(t)$, above, where \mathbf{w} solves this **higher order null space problem**, which will include a particular solution and any multiple, $k\mathbf{v}$

Bifurcation Example

1

Bifurcation Example: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

which contains a parameter α that affects the behavior of this system

We want to determine the different **qualitative behaviors** for different values of α

The eigenvalues satisfy

$$\det \begin{vmatrix} \alpha - \lambda & 2 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 - \alpha\lambda + 4 = 0$$

Thus, the eigenvalues satisfy

$$\lambda = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}$$

Bifurcation Example

2

Bifurcation Example: For

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (3)$$

The eigenvalues are $\lambda = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}$

Classifications as α varies are:

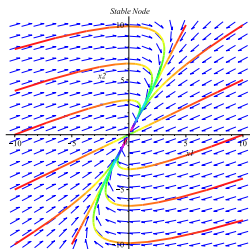
- For $\alpha < -4$, System (3) is a **Stable Node**
- For $\alpha = -4$, System (3) is a **Stable Improper Node**
- For $-4 < \alpha < 0$, System (3) is a **Stable Spiral**
- For $\alpha = 0$, System (3) is a **Center**
- For $0 < \alpha < 4$, System (3) is a **Unstable Spiral**
- For $\alpha = 4$, System (3) is a **Unstable Improper Node**
- For $\alpha > 4$, System (3) is a **Unstable Node**

Bifurcation Example

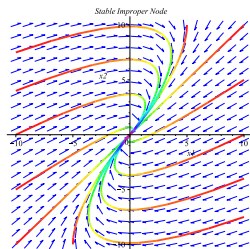
3

Bifurcation Example: Phase Portraits ($\alpha < 0$)

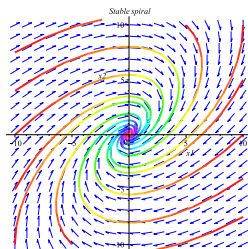
Observe a smooth transition as eigenvalues change from negative to complex with negative real part



$$\alpha = -5$$



$$\alpha = -4$$



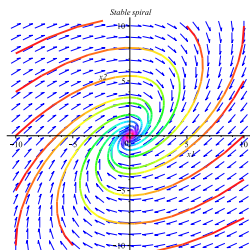
$$\alpha = -2$$

Bifurcation Example

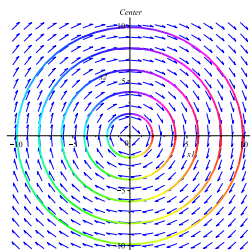
4

Bifurcation Example: Phase Portraits ($-4 < \alpha < 4$)

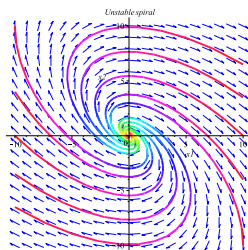
Observe the transitions as complex eigenvalues change from negative real part to positive real part - This is a significant part of a **Hopf bifurcation**



$\alpha = -2$



$\alpha = 0$



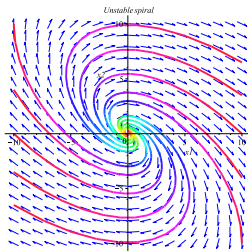
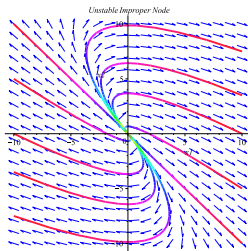
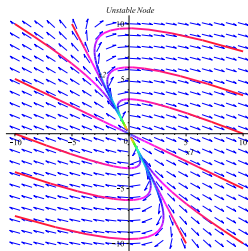
$\alpha = 2$

Bifurcation Example

5

Bifurcation Example: Phase Portraits ($\alpha > 0$)

Observe a smooth transition as eigenvalues change from complex with positive real part to positive real values

 $\alpha = 2$  $\alpha = 4$  $\alpha = 5$

Stability Diagram

Consider the system

$$\dot{\mathbf{x}} = \mathbf{J}\mathbf{x}$$

Let λ_1 and λ_2 be eigenvalues of $\mathbf{J}\mathbf{x}$

Results from Linear Algebra

give $tr(\mathbf{J}) = \lambda_1 + \lambda_2$,

$\det|\mathbf{J}| = \lambda_1 \cdot \lambda_2$, and

$$D = (j_{11} - j_{22})^2 + 4j_{12}j_{21}$$

The figure shows the

Stability Diagram for

$\dot{\mathbf{x}} = \mathbf{J}\mathbf{x}$ with axes
of $tr(\mathbf{J})$ vs $\det|\mathbf{J}|$

