Math 337 - Elementary Differential Equations

Lecture Notes – Systems of Two First Order Equations: Part A

Joseph M. Mahaffy,
⟨jmahaffy@sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://jmahaffy.sdsu.edu

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- Two Dimensional Model
- Steady State Analysis
- Eigenvalue Analysis
- Model Solution

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Introduction

Introduction

- Many applications use more than one variable
- Use techniques from Linear Algebra
- Solve basic 2-dimensional linear ordinary differential equations
  - Systems with constant coefficients
  - Find eigenvalues and eigenvectors
  - Graph **phase portraits**
  - Qualitative Analysis
- Introduce nonlinear 2D systems

Greenhouse/Rockbed
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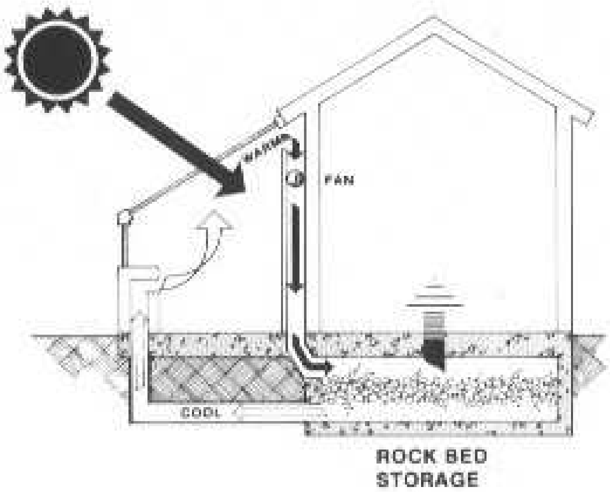
Greenhouse/Rockbed



Greenhouse/Rockbed

Greenhouse/Rockbed System

- Greenhouse heats during the day and cools at night
- Insulated bed of rocks stores and releases heat
- Automated fan pumps air from greenhouse to bed of rocks
- Greenhouse air readily heated with the sun and lost at night
- Heat capacity of rocks absorbs heat during day from hot air, then releases during night
- System can maintain a more constant temperature

Greenhouse/Rockbed

Simplified Model: Lumped system thermal analysis using Newton's Law of Cooling

Define model parameters

- m_1, m_2 Masses of Air and Rocks
- C_1, C_2 Specific heat of Air and Rocks
- A_1, A_2 Surface areas of Greenhouse and Rocks
- h_1, h_2 Heat transfer coefficients across A_1 and A_2
- T_a Temperature of air external to greenhouse

Greenhouse/Rockbed

Conservation of Energy gives

$$\begin{aligned} m_1C_1\frac{du_1}{dt} &= -h_1A_1(u_1 - T_a) - h_2A_2(u_1 - u_2) \\ m_2C_2\frac{du_2}{dt} &= -h_2A_2(u_2 - u_1) \end{aligned}$$

Can write system

$$\begin{aligned} \frac{du_1}{dt} &= -(k_1 + k_2)u_1 + k_2u_2 + K_1T_a \\ \frac{du_2}{dt} &= \varepsilon k_2u_1 - \varepsilon k_2u_2 \end{aligned}$$

with

$$k_1 = \frac{h_1A_1}{m_1C_1} \qquad k_2 = \frac{h_2A_2}{m_1C_1} \qquad \varepsilon = \frac{m_1C_1}{m_2C_2}$$

Greenhouse/Rockbed

Model Design

- Allows simulation to choose the size of rock bed and amount of airflow based on size of greenhouse
- Varying quantities and material changes coefficients
- Coefficients are known based on thermal properties of gases and building materials
- Given initial conditions

  $$u_1(0) = u_{10} \qquad \text{and} \qquad u_2(0) = u_{20}$$

  can easily simulate
- Analysis allows optimal design

Greenhouse/Rockbed

Model: Actual determining the values of the kinetic parameters for a particular greenhouse/rockbed configuration can be a very difficult problem

This is the **most important** problem in design

Suppose that we have

$$k_1 = \tfrac{7}{8} \qquad k_2 = \tfrac{3}{4} \qquad \varepsilon = \tfrac{1}{3} \qquad T_a = 16^\circ\mathrm{C}$$

Then

$$\begin{aligned} \frac{du_1}{dt} &= -\tfrac{13}{8}u_1 + \tfrac{3}{4}u_2 + 14 \\ \frac{du_2}{dt} &= \tfrac{1}{4}u_1 - \tfrac{1}{4}u_2 \end{aligned}$$

Model Analysis - Matrix Form 1

Model in Matrix Form (**Note**: We define $\frac{du_1(t)}{dt} = \dot{u}_1$.)

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} -\frac{13}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 14 \\ 0 \end{pmatrix}$$

which has the form

$$\dot{\mathbf{u}} = \mathbf{K}\mathbf{u} + \mathbf{b}$$

with initial condition

$$\mathbf{u}(0) = \mathbf{u}_0 = \begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix}$$

Model Analysis - Expectations 2

Qualitative Model Expectations

- The only energy input into the system is the environment at 16°C
- With this constant environmental temperature, expect

$$\lim_{t\to\infty}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \lim_{t\to\infty} \mathbf{u}(t) = \begin{pmatrix} 16 \\ 16 \end{pmatrix} = \mathbf{u}_e$$

- Model uses **Newton's Law of Cooling**, so expect an exponential decay toward $\mathbf{u}_e$

Model Analysis - Steady State

Model Analysis - Steady State: At **steady state**, $\dot{\mathbf{u}} = 0$

Need to solve

$$\mathbf{K}\mathbf{u} + \mathbf{b} = \mathbf{0} \qquad \text{or} \qquad \mathbf{K}\mathbf{u} = -\mathbf{b}$$

This solves the linear system

$$\begin{pmatrix} -\frac{13}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} u_{1e} \\ u_{2e} \end{pmatrix} = \begin{pmatrix} -14 \\ 0 \end{pmatrix}$$

This is readily solved by row reduction (**row reduced echelon form**)

Solve Linear System 1

Solve Linear System: Write $[\mathbf{A} : \mathbf{b}]$, so

$$\left[\begin{array}{cc:c} -\frac{13}{8} & \frac{3}{4} & -14 \\ \frac{1}{4} & -\frac{1}{4} & 0 \end{array}\right] \underset{4R_2}{\xrightarrow{-\frac{8}{13}R_1}} \left[\begin{array}{cc:c} 1 & -\frac{6}{13} & \frac{112}{13} \\ 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{cc:c} 1 & -\frac{6}{13} & \frac{112}{13} \\ 0 & -\frac{7}{13} & -\frac{112}{13} \end{array}\right] \xrightarrow{-\frac{13}{7}R_2} \left[\begin{array}{cc:c} 1 & -\frac{6}{13} & \frac{112}{13} \\ 0 & 1 & 16 \end{array}\right]$$

$$\xrightarrow{R_1 + \frac{6}{13}R_2} \left[\begin{array}{cc:c} 1 & 0 & 16 \\ 0 & 1 & 16 \end{array}\right] \quad \text{or} \quad \mathbf{u}_e = \left[\begin{array}{c} 16 \\ 16 \end{array}\right]$$

Solve Linear System 2

Solve Linear System: Linear systems are efficiently solved in **MatLab** and **Maple**

- **MatLab** - Solving equilibrium
  - Enter matrix, A, and vector, b
  - Use *linsolve* command or *inv(A)*b*
  - Augment A with b and use *rref*
- **Maple** - Solving equilibrium
  - Start *with(LinearAlgebra)* to invoke the Linear Algebra package
  - Enter matrix, A, and vector, b
  - Use *LinearSolve(A,b)* command or *Multiply(A^{-1},b)* operation
- Detailed supplemental sheets are provided

Solving the System of DEs 1

Model System satisfies

$$\dot{\mathbf{u}} = \mathbf{K}\mathbf{u} + \mathbf{b}$$

and has a **steady state solution** $\mathbf{u}(t) = \mathbf{u}_e$, where $\mathbf{K}\mathbf{u}_e = -\mathbf{b}$

Make a change of variables $\mathbf{v}(t) = \mathbf{u}(t) - \mathbf{u}_e$, then $\dot{\mathbf{v}} = \dot{\mathbf{u}}$ and

$$\dot{\mathbf{v}} = \mathbf{K}(\mathbf{v} + \mathbf{u}_e) + \mathbf{b} = \mathbf{K}\mathbf{v}$$

This **change of variables** allows considering the simpler system

$$\dot{\mathbf{v}} = \mathbf{K}\mathbf{v}$$

Solving the System of DEs 2

Model System has a **Newton's Law of Cooling**, so anticipate an exponential (decaying) solution

Try a solution of the form $\mathbf{v}(t) = \xi e^{\lambda t}$, where $\xi = [v_1, v_2]^T$ is a constant vector, so $\dot{\mathbf{v}}(t) = \lambda \xi e^{\lambda t}$

The translated **Model System** $\dot{\mathbf{v}}(t) = \mathbf{K}\mathbf{v}(t)$ becomes

$$\lambda \xi e^{\lambda t} = \mathbf{K} \xi e^{\lambda t} \qquad \text{or} \qquad \lambda \xi = \mathbf{K} \xi$$

This is the classic **eigenvalue problem**

$$(\mathbf{K} - \lambda \mathbf{I})\xi = \mathbf{0},$$

which has **eigenvalues**, λ, and associated **eigenvectors**, ξ

The solution of the **eigenvalue problem** gives the solution of the **Model System**, $\mathbf{v}(t) = \xi e^{\lambda t}$

Greenhouse Example 1

Example Model: satisfies the DE:

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} -\frac{13}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 14 \\ 0 \end{pmatrix},$$

which has the equilibrium solution

$$\mathbf{u}_e = \begin{pmatrix} 16 \\ 16 \end{pmatrix}$$

Taking $\mathbf{v}(t) = \mathbf{u}(t) - \mathbf{u}_e$, we examine the translated model

$$\begin{pmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \end{pmatrix} = \begin{pmatrix} -\frac{13}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$$

Greenhouse Example 2

Example Model: Try a solution $\mathbf{v}(t) = \xi e^{\lambda t}$ with $\xi = [\xi_1, \xi_2]^T$, so the DE can be written

$$\lambda \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} -\frac{13}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{\lambda t}$$

Dividing by $e^{\lambda t}$, we obtain the **eigenvalue problem**

$$\begin{pmatrix} -\frac{13}{8} - \lambda & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} - \lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Greenhouse Example 3

Eigenvalue Problem: Eigenvalues for the problem $(\mathbf{A} - \lambda \mathbf{I})\xi = \mathbf{0}$ solve $\det|\mathbf{A} - \lambda \mathbf{I}| = 0$, so

$$\det \begin{vmatrix} -\frac{13}{8} - \lambda & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} - \lambda \end{vmatrix} = 0$$

The **characteristic equation** is

$$\lambda^2 + \tfrac{15}{8}\lambda + \tfrac{7}{32} = 0,$$

which has solutions

$$\lambda_1 = -\tfrac{1}{8} \qquad \text{and} \qquad \lambda_2 = -\tfrac{7}{4}$$

Greenhouse Example

Eigenvalue Problem: For $\lambda_1 = -\frac{1}{8}$, we solve

$$\begin{pmatrix} -\frac{3}{2} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{8} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives a corresponding **eigenvector**, $\xi^{(1)} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$

For $\lambda_2 = -\frac{7}{4}$, we solve

$$\begin{pmatrix} \frac{1}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives a corresponding **eigenvector**, $\xi^{(2)} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$

Greenhouse Example

Solution $\mathbf{v}(t)$: The **eigenvalue problem** shows that there are two solutions to the Greenhouse example, $\dot{\mathbf{v}} = \mathbf{K}\mathbf{v}$

$$\mathbf{v}_1(t) = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} e^{-t/8} \qquad \text{and} \qquad \mathbf{v}_2(t) = \begin{pmatrix} -6 \\ 1 \end{pmatrix} e^{-7t/4}$$

along with any constant multiples of these solutions

We combine results above to obtain the **general solution**

$$\mathbf{u}(t) = c_1\mathbf{v}_1(t) + c_2\mathbf{v}_2(t) + \mathbf{u}_e = c_1 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} e^{-t/8} + c_2 \begin{pmatrix} -6 \\ 1 \end{pmatrix} e^{-7t/4} + \begin{pmatrix} 16 \\ 16 \end{pmatrix}$$

The solution exhibits the property of exponentially decaying to the steady-state solution

Greenhouse Example 6

Unique Solution: Suppose that the **rockbed** stored heat during the day, so we start with an initial condition of $u_{20}(0) = 25°\text{C}$, while the cool night air comes into the greenhouse with $u_{10}(0) = 5°\text{C}$.

To solve the IVP, we solve:

$$\mathbf{u}(0) = c_1 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -6 \\ 1 \end{pmatrix} + \begin{pmatrix} 16 \\ 16 \end{pmatrix} = \begin{pmatrix} 5 \\ 25 \end{pmatrix}$$

Equivalently, solve

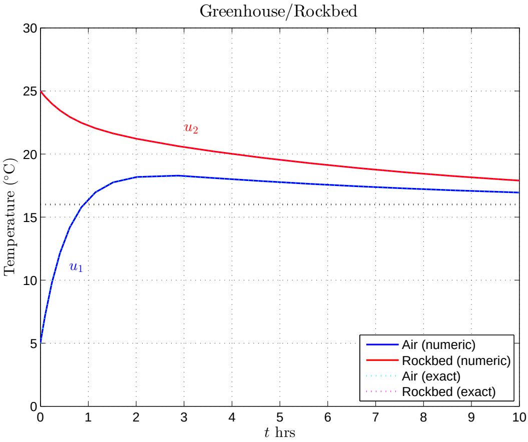
$$\begin{pmatrix} \frac{1}{2} & -6 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -11 \\ 9 \end{pmatrix} \qquad \text{or} \qquad c_1 = \frac{86}{13}, \quad c_2 = \frac{31}{13}$$

Thus, the solution to the IVP is

$$\mathbf{u}(t) = \frac{86}{13} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} e^{-t/8} + \frac{31}{13} \begin{pmatrix} -6 \\ 1 \end{pmatrix} e^{-7t/4} + \begin{pmatrix} 16 \\ 16 \end{pmatrix}$$

Greenhouse Example

Greenhouse/Rockbed Solution: Graph shows temperature in each compartment $u_1(t)$ (greenhouse) and $u_2(t)$ (rockbed)



Greenhouse Example

Greenhouse/Rockbed Solution Observations

- Both solutions tend toward the equilibrium solution of 16°C
- There is more heat capacitance in the rock (high mass), so solution changes more slowly in this compartment
- The air of the greenhouse responds more quickly (low heat capacitance)
- The air of the greenhouse heats above steady state before returning toward the equilibrium solution
- This simplified model assumes a constant external temperature of 16°C rather than the more interesting dynamics of solar power and nocturnal heat loss - significantly more complicated model

Direction Fields and Phase Portraits

Definition (Autonomous System of Differential Equations)

Let x_1 and x_2 be **state variables**, and assume that the functions, $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ are dependent only on the state variables. The **two-dimensional autonomous system of differential equations** is given by:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

Definition (Autonomous Linear System of Differential Equations)

Let x_1 and x_2 be **state variables** with $\mathbf{x} = [x_1, x_2]^T$, and assume that $\mathbf{A}$ is a constant matrix. The **autonomous linear system of differential equations** is given by:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}.$$

Direction Fields and Phase Portraits

- The **state variables**, $u_1 = u_1(t)$ and $u_2 = u_2(t)$, are **parametric equations** depending on t
- Define the vector, $\mathbf{u}(t) = u_1(t)\mathbf{i} + u_2(t)\mathbf{j}$
- The u_1u_2-plane is called the **state plane** or **phase plane**
- As t varies, the vector $\mathbf{u}(t)$ traces a curve in the phase plane called a **trajectory** or **orbit**
- An **autonomous system of differential equations** describes the dynamics of the **orbit**
- The functions, $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$, describe the slope or **direction field** in the **phase plane**
- **MatLab** and **Maple** have special routines to create **phase portraits**, which trace the **trajectories** of the **autonomous DE**

Direction Fields and Phase Portraits

Definition

Consider the **two-dimensional autonomous system of differential equations** given by:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}$$

Create the **vector field** $\mathbf{F}(x_1, x_2) = f_1(x_1, x_2)\mathbf{i} + f_2(x_1, x_2)\mathbf{j}$. The graph of the **vector field** creates the **direction field**.

Definition

A plot of **solution trajectories** for the DE with the **direction field** creates a **phase portrait**.

Phase portraits are critical tools for the **qualitative behavior** of a system of autonomous differential equations.

Greenhouse Example Revisited

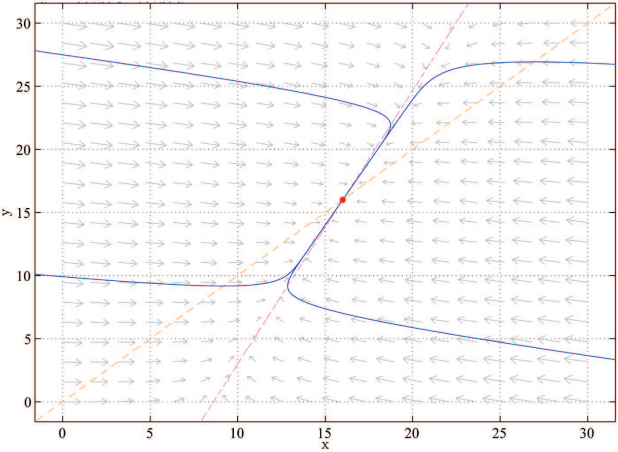
- The greenhouse example satisfied the DE

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} -\frac{13}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 14 \\ 0 \end{pmatrix},$$

- First we found an **equilibrium**, which is a point where the **direction field** is **zero**
- Useful to find **nullclines**, where $\dot{u}_1 = 0$ or $\dot{u}_2 = 0$
- The line $-\frac{13}{8}u_1 + \frac{3}{4}u_2 = -14$ has $\dot{u}_1 = 0$, while the line $\frac{1}{4}u_1 - \frac{1}{4}u_2 = 0$ has $\dot{u}_2 = 0$
- Intersection of these **nullclines** gives the **equilibrium**
- Next slide shows **phase portrait** produced by **MatLab's** *pplane8* (created by John Polking at Rice University)

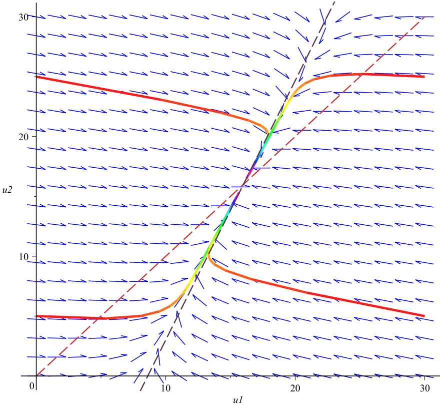
Greenhouse Example Revisited

Greenhouse/Rockbed Phase Portrait: Graph produced by *pplane8* in **MatLab**



Greenhouse Example Revisited

Greenhouse/Rockbed Phase Portrait: Graph produced by *DEplot* in **Maple**



MatLab Summary

- **MatLab hyperlink** provides detailed instructions for this section
- **MatLab**
  - MatLab is well-designed to **solve** linear systems, *linsolve*, for Equilibria
  - MatLab readily finds eigenvalues and eigenvectors, *eig*, for the eigenvalue problem needed to solve systems of linear DEs
  - Numerical solutions use package like *ode23*
  - Nonlinear equations can have equilibria found with *fsolve*
  - Phase portraits and direction fields are graphed using *pplane* from Rice University

Maple Summary

- **Maple hyperlink** provide detailed instructions for this section
- **Maple**
  - Maple has a *LinearAlgebra* package
  - This package has commands *LinearSolve*, *Eigenvectors*, and many more for managing linear systems of DEs
  - Exact solutions of linear systems are found with *dsolve*
  - Phase portraits and direction fields are graphed with the package *DEtools* and the program *DEplot*