Math 337 - Elementary Differential Equations

Lecture Notes – Laplace Transforms: Part B

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Outline

Inverse Laplace Transforms

Theorem (Inverse Laplace Transform)

If $f(t)$ and $g(t)$ are piecewise continuous and have exponential order with exponent a on $[0,\infty)$ and $F = G$, where $F = \mathcal{L}[f]$ and $G = \mathcal{L}[g]$, then $f(t) = g(t)$ at all points where both f and g are continuous. In particular, f and g are continuous on $[0,\infty)$, then $f(t) = g(t)$ for all $t \in [0,\infty)$.

The functions may disagree at points of **discontinuity**

Definition (Inverse Laplace Transform)

If $f(t)$ is piecewise continuous and has exponential order with exponent a on $[0,\infty)$ and $\mathcal{L}[f(t)] = F(s)$, then we call f the **inverse Laplace transform** of F, and denote it by

$$f(t) = \mathcal{L}^{-1}[F(s)].$$

Linearity of Inverse Laplace Transforms

Theorem (Linearity of Inverse Laplace Transform)

Assume that $f_1 = \mathcal{L}^{-1}[F_1]$ *and* $f_2 = \mathcal{L}^{-1}[F_2]$ *are piecewise continuous and has exponential of order with exponent* a *on* $[0,\infty)$*. Then for any constants* c_1 *and* c_2*,*

$$\mathcal{L}^{-1}[c_1F_1 + c_2F_2] = c_1\mathcal{L}^{-1}[F_1] + c_2\mathcal{L}^{-1}[F_2] = c_1f_1 + c_2f_2.$$

Example: Find $\mathcal{L}^{-1}\left[\frac{2}{(s-3)^4} + \frac{12}{s^2+16} + \frac{5(s+2)}{s^2+4s+5}\right]$.

Rewrite as

$$\frac{1}{3}\mathcal{L}^{-1}\left[\frac{3!}{(s-3)^4}\right] + 3\mathcal{L}^{-1}\left[\frac{4}{s^2+16}\right] + 5\mathcal{L}^{-1}\left[\frac{(s+2)}{(s+2)^2+1}\right]$$

With **Exponential Shift Theorem**

$$\frac{1}{3}e^{3t}t^3 + 3\sin(4t) + 5e^{-2t}\cos(t)$$

Example: DE with Laplace Transform 1

Example: Consider the **initial value problem**:

$$y'' + y = e^{-t}\cos(2t) \qquad \text{with} \quad y(0) = 2, \quad y'(0) = 1$$

Let $Y(s) = \mathcal{L}[y(t)]$, then taking **Laplace transforms** gives

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{s+1}{(s+1)^2 + 4}$$

or

$$(s^2+1)Y(s) = 2s + 1 + \frac{s+1}{(s+1)^2+4}$$

Equivalently,

$$Y(s) = \frac{2s+1}{s^2+1} + \frac{s+1}{(s^2+1)((s+1)^2+4)}$$

Example: DE with Laplace Transform 2

Example: Since

$$Y(s) = \frac{2s+1}{s^2+1} + \frac{s+1}{(s^2+1)((s+1)^2+4)},$$

and the first term is already in simplest form, **partial fractions decomposition** gives

$$\frac{s+1}{(s^2+1)((s+1)^2+4)} = \frac{As+B}{s^2+1} + \frac{C(s+1)+D\cdot 2}{(s+1)^2+4}.$$

It follows that

$$s+1 = (As+B)(s^2+2s+5) + (C(s+1)+2D)(s^2+1)$$

Let $s = i$, then

$$1+i = (B+Ai)(4+2i) = (4B-2A) + i(4A+2B)$$

Example: DE with Laplace Transform 3

Example: Equating the real and imaginary parts of the previous equation give:

$$-2A + 4B = 1 \qquad \text{and} \qquad 4A + 2B = 1$$

Solving the **linear equations** gives $A = \frac{1}{10}$ and $B = \frac{3}{10}$

Since

$$s + 1 = (As + B)(s^2 + 2s + 5) + (C(s + 1) + 2D)(s^2 + 1),$$

the cubic (s^3) terms give $0 = A + C$ or $C = -\frac{1}{10}$.

The constant terms give

$$1 = 5B + C + 2D \qquad \text{or} \qquad D = -\frac{1}{5}$$

Example: DE with Laplace Transform 4

Example: Thus,

$$
\begin{aligned}
Y(s) &= \frac{2s+1}{s^2+1} + \frac{s+1}{(s^2+1)((s+1)^2+4)}, \\
Y(s) &= \frac{2s+1}{s^2+1} + \frac{s/10}{s^2+1} + \frac{3/10}{s^2+1} - \frac{(s+1)/10}{(s+1)^2+4} - \frac{2/5}{(s+1)^2+4} \\
Y(s) &= \frac{\left(\frac{21}{10}\right)s}{s^2+1} + \frac{\left(\frac{13}{10}\right)\cdot 1}{s^2+1} - \frac{\left(\frac{1}{10}\right)(s+1)}{(s+1)^2+4} - \frac{\left(\frac{1}{5}\right)\cdot 2}{(s+1)^2+4}
\end{aligned}
$$

This last line allows easy application of the **Inverse Laplace Transform** to obtain the solution

$$
y(t) = \frac{21}{10}\cos(t) + \frac{13}{10}\sin(t) - \frac{1}{10}e^{-t}\cos(2t) - \frac{1}{5}e^{-t}\sin(2t).
$$

Laplace for Systems of DEs 1

Laplace for Systems of Differential Equations: Consider

$$\begin{aligned} \dot{y}_1 &= a_{11}y_1 + a_{12}y_2 + f_1(t), \qquad y_1(0) = y_{10} \\ \dot{y}_2 &= a_{21}y_1 + a_{22}y_2 + f_2(t), \qquad y_2(0) = y_{20} \end{aligned}$$

Taking **Laplace transforms** gives

$$\begin{aligned} sY_1 - y_1(0) &= a_{11}Y_1 + a_{12}Y_2 + F_1(s), \\ sY_2 - y_2(0) &= a_{21}Y_1 + a_{22}Y_2 + F_2(s), \end{aligned}$$

which can be written

$$\begin{aligned} (s - a_{11})Y_1 - a_{12}Y_2 &= y_1(0) + F_1(s), \\ -a_{21}Y_1 + (s - a_{22})Y_2 &= y_2(0) + F_2(s). \end{aligned}$$

Or in **matrix form**

$$(s\mathbf{I} - \mathbf{A})\mathbf{Y} = \mathbf{y}_0 + \mathbf{F}(s)$$

Laplace for Systems of DEs 2

Laplace for Systems of Differential Equations: The **matrix form** is

$$(s\mathbf{I} - \mathbf{A})\mathbf{Y} = \mathbf{y}_0 + \mathbf{F}(s),$$

which is readily solved as

$$\mathbf{Y} = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}_0 + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{F}(s)$$

The inverse satisfies

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{|s\mathbf{I} - \mathbf{A}|}\begin{pmatrix} s - a_{22} & a_{12} \\ a_{21} & s - a_{11} \end{pmatrix},$$

where

$$|s\mathbf{I} - \mathbf{A}| = s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21}$$

is the **characteristic polynomial**

This is not hard to solve algebraically, but the inverse Laplace transform may be messy

Example: Laplace for System 1

Example: Consider the nonhomogeneous system

$$\dot{\mathbf{y}} = \begin{pmatrix} -4 & -1 \\ 1 & -2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 2e^t \\ \sin(2t) \end{pmatrix}, \qquad \mathbf{y}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Taking **Laplace transforms** gives

$$\begin{aligned} sY_1 - 1 &= -4Y_1 - Y_2 + \frac{2}{s-1} \\ sY_2 - 2 &= Y_1 - 2Y_2 + \frac{2}{s^2+4} \end{aligned}$$

Equivalently,

$$\begin{pmatrix} s+4 & 1 \\ -1 & s+2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 + \frac{2}{s-1} \\ 2 + \frac{2}{s^2+4} \end{pmatrix}$$

Example: Laplace for System 2

Example: If

$$(s\mathbf{I}-\mathbf{A}) = \begin{pmatrix} s+4 & 1 \\ -1 & s+2 \end{pmatrix}, \qquad \mathbf{y}_0 + \mathbf{F}(s) = \begin{pmatrix} 1+\frac{2}{s-1} \\ 2+\frac{2}{s^2+4} \end{pmatrix},$$

then the solution is

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = (s\mathbf{I}-\mathbf{A})^{-1}(\mathbf{y}_0+\mathbf{F}(s)),$$

where

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{(s+3)^2}\begin{pmatrix} s+2 & -1 \\ 1 & s+4 \end{pmatrix}$$

The expressions for $Y_1(s)$ and $Y_2(s)$ are fairly complex, so we show how **Maple** can help solve these expressions into a form, which readily has an **inverse Laplace transform**

Example: Laplace for System 3

Example: From the previous page, it is easy to see that

$$\begin{aligned} Y_1(s) &= \frac{s}{(s+3)^2} + \frac{2(s+2)}{(s-1)(s+3)^2} - \frac{2}{(s^2+4)(s+3)^2} \\ Y_2(s) &= \frac{2s+9}{(s+3)^2} + \frac{2}{(s-1)(s+3)^2} + \frac{2(s+4)}{(s^2+4)(s+3)^2} \end{aligned}$$

One can perform **Partial Fractions Decomposition** on these expressions, which is a very messy process.

We demonstrate how this can be done with **Maple**, which can readily perform both **Partial Fractions Decompositions** and **inverse Laplace transforms**

A **Special Maple Sheet** is provided along with a complete solution using **Maple**

Example: Laplace for System 4

Example: Maple gives the **Laplace transform**

$$
\begin{aligned}
Y_1(s) &= \frac{\frac{3}{8}}{s-1} + \frac{\frac{12s-10}{169}}{s^2+4} - \frac{\frac{69}{26}}{(s+3)^2} + \frac{\frac{749}{1352}}{s+3} \\
Y_2(s) &= \frac{\frac{1}{8}}{s-1} + \frac{\frac{88-38s}{169}}{s^2+4} + \frac{\frac{69}{26}}{(s+3)^2} + \frac{\frac{2839}{1352}}{s+3}
\end{aligned}
$$

The **inverse Laplace transform** gives the **solution**

$$
\begin{aligned}
y_1(t) &= \frac{3}{8}e^t + \frac{12}{169}\cos(2t) - \frac{5}{169}\sin(2t) - \frac{69}{26}te^{-3t} + \frac{749}{1352}e^{-3t} \\
y_2(t) &= \frac{1}{8}e^t + \frac{44}{169}\sin(2t) - \frac{38}{169}\cos(2t) + \frac{69}{26}te^{-3t} + \frac{2839}{1352}e^{-3t}
\end{aligned}
$$

Discontinuous Functions

Unit Step function or **Heaviside function** satisfies

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

The translated version of the **Unit Step function** by c units is

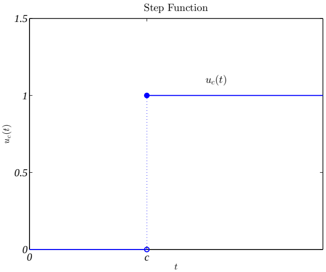
$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c, \end{cases}$$

which represents a switch **turning on** at $t = c$

An **indicator function**, which is **on** for $c \leq t < d$, satisfies

$$u_{cd}(t) = u_c(t) - u_d(t) = \begin{cases} 0, & t < c \quad \text{or} \quad t \geq d, \\ 1, & c \leq t < d, \end{cases}$$

Laplace Transform of Step Function



Laplace Transform of Step Function;

$$
\begin{aligned}
\mathcal{L}[u_c(t)] &= \int_0^\infty e^{-st}u_c(t)dt = \int_c^\infty e^{-st}dt = \lim_{A\to\infty}\int_c^A e^{-st}dt \\
&= \lim_{A\to\infty}\left(\frac{e^{-cs}}{s} - \frac{e^{-sA}}{s}\right) = \frac{e^{-cs}}{s}, \qquad s > 0
\end{aligned}
$$

Laplace Transform of Step Function

Theorem

If $F(s) = \mathcal{L}[f(t)]$ exists for $s > a \geq 0$, and if c is a nonnegative constant, then

$$\mathcal{L}[u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)] = e^{-cs}F(s), \qquad s > a.$$

Conversely, if $f(t) = \mathcal{L}^{-1}[F(s)]$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}[e^{-cs}F(s)].$$

This theorem states that the translation of $f(t)$ a distance c in the positive t direction corresponds to the multiplication of $F(s)$ by e^{-cs}

Example with Step Function

Example with Step Function: Consider the following **initial value problem**:

$$y'' + 2y' + 5y = u_2(t) - u_5(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Take **Laplace transforms** and obtain

$$(s^2 + 2s + 5)Y(s) = \frac{e^{-2s}}{s} - \frac{e^{-5s}}{s}$$

This rearranges to

$$Y(s) = \frac{e^{-2s} - e^{-5s}}{s(s^2 + 2s + 5)}$$

Partial fraction decomposition gives

$$\frac{1}{s(s^2 + 2s + 5)} = \frac{A}{s} + \frac{B(s+1) + 2C}{(s+1)^2 + 4}$$

Example with Step Function 2

Example with Step Function: Partial fraction decomposition gives

$$1 = A\left((s+1)^2+4\right) + (B(s+1)+2C)\,s$$

With $s = 0$, $1 = 5A$ or $A = \frac{1}{5}$

The s^2 coefficient gives $0 = A + B$, so $B = -\frac{1}{5}$

The s^1 coefficient gives $0 = 2A + B + 2C$, so $C = -\frac{1}{10}$

Thus,

$$Y(s) = \left(\frac{\frac{1}{5}}{s} - \frac{\frac{1}{5}(s+1)+\frac{2}{10}}{(s+1)^2+4}\right)\left(e^{-2s}-e^{-5s}\right)$$

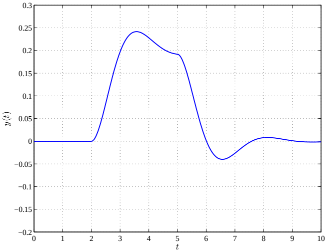
Example with Step Function

Example with Step Function: With the **Laplace transform**

$$Y(s) = \left(\frac{\frac{1}{5}}{s} - \frac{\frac{1}{5}(s+1) + \frac{2}{10}}{(s+1)^2 + 4} \right) \left(e^{-2s} - e^{-5s}\right)$$

The theorem for step functions allows the **inverse Laplace transform** yielding

$$\begin{aligned} y(t) &= \frac{u_2(t)}{10}\left(2 - 2e^{-(t-2)}\cos(2(t-2)) - e^{-(t-2)}\sin(2(t-2))\right) \\ &\quad -\frac{u_5(t)}{10}\left(2 - 2e^{-(t-5)}\cos(2(t-5)) - e^{-(t-5)}\sin(2(t-5))\right) \end{aligned}$$



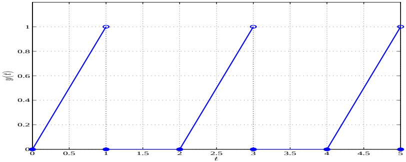
Periodic Functions

Definition

A function f is said to be **periodic with period $T > 0$** if

$$f(t+T) = f(t)$$

for all t in the domain of f.



A **sawtooth** waveform

$$f(t) = \begin{cases} t, & 0 \le t < 1, \\ 0, & 1 \le t < 2. \end{cases} \qquad \text{and } f(t) \text{ has period } 2$$

Periodic and Window functions

Consider a **periodic function** $f(t)$. Define the **window function**, $f_T(t)$, as follows:

$$f_T(t) = f(t)\left[1 - u_T(t)\right] = \begin{cases} f(t), & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

The **Laplace transform** $F_T(s)$ satisfies:

$$F_T(s) = \int_0^\infty e^{-st} f_T(t)dt = \int_0^T e^{-st} f_T(t)dt.$$

The **window function** specifies values of $f(t)$ over a single period

This can be replicated k periods to the right as

$$f_T(t - kT)u_{kT}(t) = \begin{cases} f(t - kT), & kT \le t \le (k+1)T \\ 0, & \text{otherwise} \end{cases}$$

Laplace for Periodic Functions

By summing n time shifted replications of the **window function**, $f_T(t-kT)u_{kT}(t)$, $k = 0,\dots,n-1$, gives $f_{nT}(t)$, the periodic extension of $f_T(t)$ to the interval $[0, nT]$,

$$f_{nT}(t) = \sum_{k=0}^{n-1} f_T(t-kT)u_{kT}(t)$$

Theorem

If f is periodic with period T and is piecewise continuous on $[0,T]$, then

$$\mathcal{L}[f(t)] = \frac{F_T(s)}{1-e^{-sT}} = \frac{\int_0^T e^{-st} f(t)dt}{1-e^{-sT}}.$$

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Laplace for Periodic Functions

Proof: From our earlier theorem, we have for each $k \geq 0$,

$$\mathcal{L}[f_T(t-kT)u_{kT}(t)] = e^{-kTs}\mathcal{L}[f_T(t)] = e^{-kTs}F_T(s).$$

By linearity of $\mathcal{L}$, the **Laplace transform** of f_{nT} is

$$\begin{aligned} F_{nT}(s) &= \int_0^{nT} e^{-st}f(t)dt = \sum_{k=0}^{n-1}\mathcal{L}[f_T(t-kT)u_{kT}(t)] \\ &= \sum_{k=0}^{n-1} e^{-kTs}F_T(s) = F_T(s)\sum_{k=0}^{n-1}\left(e^{-Ts}\right)^k = F_T(s)\frac{1-\left(e^{-Ts}\right)^n}{1-e^{-sT}}. \end{aligned}$$

The last term comes from summing a geometric series. With $e^{-sT} < 1$,

$$F(s) = \lim_{n->\infty}\int_0^{nT} e^{-st}dt = \lim_{n->\infty} F_T(s)\frac{1-\left(e^{-Ts}\right)^n}{1-e^{-sT}} = \frac{F_T(s)}{1-e^{-sT}}$$

Sawtooth Function

Return to **sawtooth** waveform

$$f(t) = \begin{cases} t, & 0 \le t < 1, \\ 0, & 1 \le t < 2. \end{cases} \qquad \text{and } f(t) \text{ has period } 2$$

The **theorem** for the **Laplace transform** of **periodic function** gives

$$\mathcal{L}[f(t)] = \frac{\int_0^2 e^{-st} f(t) dt}{1 - e^{-2s}}$$

But

$$\int_0^2 e^{-st} f(t) dt = \int_0^1 t e^{-st} dt = \frac{1 - s e^{-s} - e^{-s}}{s^2},$$

so

$$\mathcal{L}[f(t)] = \frac{1 - s e^{-s} - e^{-s}}{s^2 \left(1 - e^{-2s}\right)}$$

IVP with Periodic Forcing Function 1

Example: Consider the following initial value problem:

$$y'' + 4y = f(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

with the **square** waveform as the **periodic forcing function**:

$$f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & 1 \le t < 2. \end{cases} \qquad \text{and } f(t) \text{ has period 2}$$

The **theorem** for the **Laplace transform** of **square** waveform gives

$$\mathcal{L}[f(t)] = \frac{\int_0^2 e^{-st} f(t)dt}{1 - e^{-2s}}$$

But

$$\int_0^2 e^{-st} f(t)dt = \int_0^1 e^{-st} dt = \frac{1 - e^{-s}}{s},$$

so

$$\mathcal{L}[f(t)] = \frac{1 - e^{-s}}{s\left(1 - e^{-2s}\right)} = \frac{1}{s\left(1 + e^{-s}\right)}$$

IVP with Periodic Forcing Function 2

Example: Taking the **Laplace transform** of the **IVP** with $\mathcal{L}[y(t)] = Y(s)$, we have:

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{s\,(1+e^{-s})}$$

Thus,

$$Y(s) = \frac{1}{s(s^2+4)\,(1+e^{-s})}$$

Partial fractions decomposition gives

$$\frac{1}{s(s^2+4)} = \frac{1/4}{s} - \frac{s/4}{s^2+4},$$

while

$$\frac{1}{1+e^{-s}} = \frac{1}{1-(-e^{-s})} = 1 - e^{-s} + e^{-2s} - ... + (-1)^n e^{-ns} +$$

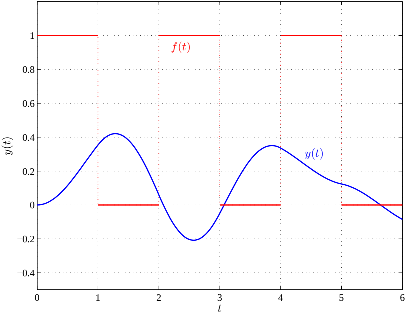
IVP with Periodic Forcing Function 3

Example: So

$$Y(s) = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\sum_{k=0}^{\infty}(-1)^k e^{-ks}$$

Taking the **inverse Laplace transform** gives:

$$y(t) = \frac{1}{4}\left(1-\cos(2t)\right) + \frac{1}{4}\sum_{k=1}^{\infty}(-1)^k u_k(t)\left(1-\cos(2(t-k))\right)$$



Impulse Function

Impulse Function: Some applications have phenomena of an **impulsive nature**, *e.g.*, a large magnitude force over a very short time

$$ay'' + by' + cy = g(t),$$

where $g(t)$ is very large for $t \in [t_0, t_0 + \varepsilon)$ and is otherwise zero

Example: Let $t_0 = 0$ be a real number and ε be a small positive constant

Suppose $t_0 = 0$ and $g(t) = I_0 \delta_\varepsilon(t)$, where

$$\delta_\varepsilon(t) = \frac{u_0(t) - u_\varepsilon(t)}{\varepsilon} = \begin{cases} \frac{1}{\varepsilon}, & 0 \le t < \varepsilon, \\ 0, & t < 0 \quad \text{or} \quad t \ge \varepsilon. \end{cases}$$

Consider the **mass-spring** system $m = 1$, $\gamma = 0$, and $k = 1$

$$y'' + y = I_0 \delta_\varepsilon(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Impulse Function - Example

Example: With $\delta_\varepsilon(t) = \frac{u_0(t)-u_\varepsilon}{\varepsilon}$, the **Laplace transform** is easy for

$$y'' + y = I_0\delta_\varepsilon(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

It satisfies

$$(s^2+1)Y(s) = \frac{I_0}{\varepsilon}\left(\frac{1-e^{-\varepsilon s}}{s}\right),$$

so

$$Y(s) = \frac{I_0}{\varepsilon}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\left(1-e^{-\varepsilon s}\right)$$

The **inverse Laplace transform** gives

$$y_\varepsilon(t) = \frac{I_0}{\varepsilon}\left(u_0(t)(1-\cos(t)) - u_\varepsilon(t)(1-\cos(t-\varepsilon))\right)$$

Impulse Function - Example

Example: Since

$$y_\varepsilon(t) = \frac{I_0}{\varepsilon}\left(u_0(t)(1-\cos(t)) - u_\varepsilon(t)(1-\cos(t-\varepsilon))\right),$$

equivalently:

$$y_\varepsilon(t) = \begin{cases} 0, & t < 0, \\ \frac{I_0}{\varepsilon}(1-\cos(t)) & 0 \le t < \varepsilon, \\ \frac{I_0}{\varepsilon}(\cos(t-\varepsilon) - \cos(t)) & t \ge \varepsilon. \end{cases}$$

The limiting case is

$$y_0(t) = \lim_{\varepsilon \to 0} y_\varepsilon(t) = u_0(t) I_0 \sin(t) = \begin{cases} 0, & t < 0, \\ I_0 \sin(t), & t \ge 0. \end{cases}$$

Unit Impulse Function

Unit Impulse Function: Rather than using the definition of $\delta_\varepsilon(t-t_0)$ to model an impulse, then take the limit as $\varepsilon \to 0$, we define an idealized **unit Impulse Function,** δ

- The "function" δ imparts an impulse of magnitude **1** at $t = t_0$, but is **zero** for all other values of t
- **Properties of** $\delta(t-t_0)$
  1. Limiting behavior:
  $$\delta(t-t_0) = \lim_{\varepsilon\to 0} \delta_\varepsilon(t-t_0) = 0$$
  2. If f is continuous for $t \in [a,b)$ and $t_0 \in [a,b)$, then
  $$\int_a^b f(t)\delta(t-t_0)dt = \lim_{\varepsilon\to 0}\int_a^b f(t)\delta_\varepsilon(t-t_0)dt = f(t_0).$$

$\delta(t - t_0)$

Dirac delta function, $\delta(t-t_0)$: This is not an ordinary function in elementary calculus, and it satisfies:

$$\int_a^b \delta(t-t_0)dt = \begin{cases} 1, & \text{if} \quad t_0 \in [a,b), \\ 0, & \text{if} \quad t_0 \notin [a,b). \end{cases}$$

The **Laplace transform of** $\delta(t-t_0)$ follows easily:

$$\mathcal{L}[\delta(t-t_0)] = \int_0^\infty e^{-st}\delta(t-t_0)dt = e^{-st_0}$$

Note: $\mathcal{L}[\delta(t)] = 1$.

The **delta function** is the **symbolic derivative** of the **Heaviside function**, so

$$\delta(t-t_0) = u'(t-t_0)$$

This is rigorously true in the theory of **generalized functions** or **distributions**

Example for $\delta(t - t_0)$ 1

Example: Consider the initial value problem:

$$y'' + 2y' + 2y = \frac{t}{\pi}\delta(t - \pi), \qquad y(0) = 0, \quad y'(0) = 1$$

The **Laplace transform** of the forcing function is

$$F(s) = \int_0^\infty e^{-st}\left(\frac{t}{\pi}\delta(t - \pi)\right) dt = e^{-\pi s}$$

It follows that the **Laplace transform** of the IVP is

$$s^2Y(s) - 1 + 2sY(s) + 2Y(s) = e^{-\pi s},$$

so

$$Y(s) = \frac{1 + e^{-\pi s}}{(s + 1)^2 + 1}$$

Example for $\delta(t - t_0)$ 2

Example: Since $Y(s) = \frac{1+e^{-\pi s}}{(s+1)^2+1}$, the **inverse Laplace transform** satisfies:

$$y(t) = e^{-t}\sin(t) + u_\pi(t)e^{-(t-\pi)}\sin(t-\pi)$$