Math 337 - Elementary Differential Equations

Lecture Notes – Laplace Transforms: Part A

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Outline

Integral Transforms

Integral Transform: This is a relation

$$F(s) = \int_{\alpha}^{\beta} K(t,s) f(t) dt,$$

which takes a given function $f(t)$ and outputs another function $F(s)$

The function $K(t,s)$ is the integral **kernel** of the transform, and the function $F(s)$ is the **transform** of $f(t)$

- **Integral Transforms** allow one to find solutions of problems (usually involving differentiation) through algebraic methods
- Properties of the **Integral Transform** allow manipulation of the function in the transformed to an easier expression, which can be inverted to find a **solution**

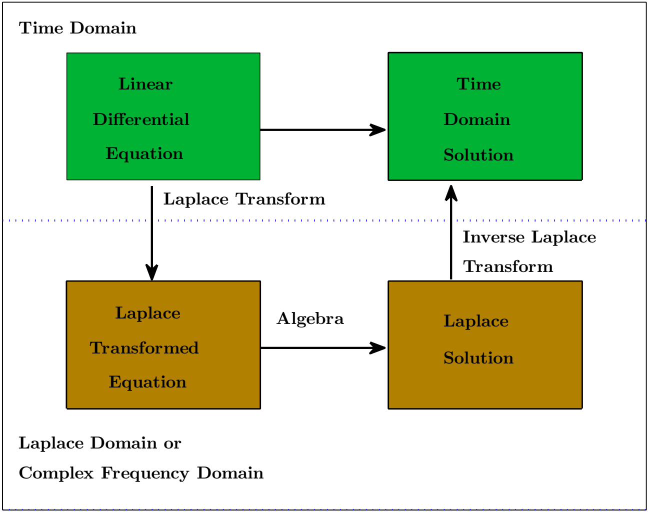
Integral Transforms

Integral Transforms

- There are many **Integral Transforms** for different problems
  - For **Partial Differential Equations** and working on the spatial domain, the **Fourier transform** is most common and defined by
$$\mathcal{F}(u) = \int_{-\infty}^{+\infty} e^{-2\pi iux} f(x)dx.$$
  - For **Ordinary Differential Equations** and working on the time domain, the **Laplace transform** is most common and defined by
$$\mathcal{L}(s) = \int_{0}^{\infty} e^{-st} f(t)dt.$$

Laplace Transforms

Laplace Transforms



Improper Integral

Improper Integral: This should be a review

The **improper integral** is defined on an **unbounded interval** and is defined

$$\int_{\alpha}^{\infty} f(t)dt = \lim_{A\to\infty} \int_{\alpha}^{A} f(t)dt,$$

where A is a positive real number

If the limit as $A \to \infty$ exists, then the **improper integral** is said to **converge** to the limiting value

Otherwise, the **improper integral** is said to **diverge**

Example: Let $f(t) = e^{ct}$ with c nonzero constant. Then

$$\int_0^{\infty} e^{ct}dt = \lim_{A\to\infty} \int_0^{A} e^{ct}dt = \lim_{A\to\infty} \left.\frac{e^{ct}}{c}\right|_0^A = \lim_{A\to\infty} \frac{1}{c}\left(e^{cA} - 1\right)$$

This **converges** for $c < 0$ and **diverges** for $c \geq 0$.

Laplace Transform

Definition (Laplace Transform)

Let f be a function on $[0,\infty)$. The **Laplace transform** of f is the function F defined by the integral,

$$F(s) = \int_0^\infty e^{-st} f(t)dt.$$

The domain of $F(s)$ is the set of all values of s for which this integral **converges**. The **Laplace transform** of f is denoted by both F and $\mathcal{L}$.

Convention uses s as the independent variable and capital letters for the transformed functions:

$\mathcal{L}[f] = F$	$\mathcal{L}[y] = Y$	$\mathcal{L}[x] = X$
$\mathcal{L}[f](s) = F(s)$	$\mathcal{L}[y](s) = Y(s)$	$\mathcal{L}[x](s) = X(s)$

Examples: Laplace Transform

Example 1: Let $f(t) = 1, t \geq 0$. The **Laplace transform** satisfies:

$$\mathcal{L}[1] = \int_0^\infty e^{-st} dt = -\lim_{A \to \infty} \frac{e^{-st}}{s}\bigg|_0^A = -\lim_{A \to \infty} \left(\frac{e^{-sA}}{s} - \frac{1}{s} \right) = \frac{1}{s}, \qquad s > 0.$$

Example 2: Let $f(t) = e^{at}, t \geq 0$. The **Laplace transform** satisfies:

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}, \qquad s > a.$$

Example 3: Let $f(t) = e^{(a+bi)t}, t \geq 0$. The **Laplace transform** satisfies:

$$\mathcal{L}[e^{(a+bi)t}] = \int_0^\infty e^{-st} e^{(a+bi)t} dt = \int_0^\infty e^{-(s-a-bi)t} dt = \frac{1}{s-a-bi},$$

$$s > a.$$

Laplace Transform - Linearity

The **Laplace transform** is a **linear operator**

Theorem (Linearity of Laplace Transform)

*Suppose the f_1 and f_2 are two functions where **Laplace transforms** exist for $s > a_1$ and $s > a_2$, respectively. Let c_1 and c_2 be real or complex numbers. Then for $s > \max\{a_1, a_2\}$,*

$$\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1 \mathcal{L}[f_1(t)] + c_2 \mathcal{L}[f_2(t)].$$

The **proof** uses the linearity of integrals.

Examples: Laplace Transform

Example 4: Let $f(t) = \sin(at), t \ge 0$. But

$$\sin(at) = \frac{1}{2i}\left(e^{iat} - e^{-iat}\right).$$

By linearity, the **Laplace transform** satisfies:

$$\mathcal{L}[\sin(at)] = \frac{1}{2i}\left(\mathcal{L}[e^{iat}] - \mathcal{L}[e^{-iat}]\right) = \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) = \frac{a}{s^2+a^2},$$

$$s > 0.$$

Example 5: Let $f(t) = 2 + 5e^{-2t} - 3\sin(4t), t \ge 0$. By linearity, the **Laplace transform** satisfies:

$$\begin{aligned} \mathcal{L}[2 + 5e^{-2t} - 3\sin(4t)] &= 2\mathcal{L}[1] + 5\mathcal{L}[e^{-2t}] - 3\mathcal{L}[\sin(4t)] \\ &= \frac{2}{s} + \frac{5}{s+2} - \frac{12}{s^2+16}, \qquad s > 0. \end{aligned}$$

Examples: Laplace Transform

Example 6: Let $f(t) = t\cos(at), t \geq 0$. The **Laplace transform** satisfies:

$$\mathcal{L}[t\cos(at)] = \int_0^\infty e^{-st} t\cos(at)dt = \frac{1}{2}\int_0^\infty \left(te^{-(s-ia)t} + te^{-(s+ia)t}\right) dt.$$

Integration by parts gives

$$\int_0^\infty te^{-(s-ia)t}dt = \left[\frac{te^{-(s-ia)t}}{s-ia} + \frac{e^{-(s-ia)t}}{(s-ia)^2}\right]_0^\infty = \frac{1}{(s-ia)^2}, \qquad s > 0.$$

Similarly,

$$\int_0^\infty te^{-(s+ia)t}dt = \frac{1}{(s+ia)^2}, \qquad s > 0.$$

Thus,

$$\mathcal{L}[t\cos(at)] = \frac{1}{2}\left[\frac{1}{(s-ia)^2} + \frac{1}{(s+ia)^2}\right] = \frac{s^2-a^2}{(s^2+a^2)^2}, \qquad s > 0.$$

Piecewise Continuous Functions

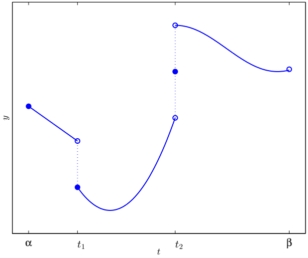
Definition (Piecewise Continuous)

A function f is said to be a **piecewise continuous** on an interval $\alpha \leq t \leq \beta$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < ... < t_n = \beta$ so that:

1. f is continuous on each subinterval $t_{i-1} < t < t_i$, and
2. f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

The figure to the right shows the graph of a **piecewise continuous** function defined for $t \in [\alpha, \beta)$ with **jump discontinuities** at $t = t_1$ and t_2.

It is **continuous** on the subintervals (α, t_1), (t_1, t_2), and (t_2, β).



Examples: Laplace Transform

Example 7: Define the **piecewise continuous** function

$$f(t) = \begin{cases} e^{2t}, & 0 \le t < 1, \\ 4, & 1 \le t. \end{cases}$$

The **Laplace transform** satisfies:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t) dt = \int_0^1 e^{-st} e^{2t} dt + \int_1^\infty e^{-st} \cdot 4 dt \\ &= \int_0^1 e^{-(s-2)t} dt + 4 \lim_{A \to \infty} \int_1^A e^{-st} dt \\ &= -\left. \frac{e^{-(s-2)t}}{s-2} \right|_{t=0}^{1} - 4 \lim_{A \to \infty} \left. \frac{e^{-st}}{s} \right|_{t=1}^{A} \\ &= \frac{1}{s-2} - \frac{e^{-(s-2)}}{s-2} + 4 \frac{e^{-s}}{s}, \qquad s > 0, s \ne 2. \end{aligned}$$

Existence of Laplace Transform

Definition (Exponential Order)

A function $f(t)$ is of **exponential order** (as $t \to +\infty$) if there exist real constants $M \geq 0$, $K > 0$, and a, such that

$$|f(t)| \leq Ke^{at},$$

when $t \geq M$.

Examples:

- $f(t) = \cos(\alpha t)$ satisfies being of **exponential order** with $M = 0$, $K = 1$, and $a = 0$
- $f(t) = t^2$ satisfies being of **exponential order** with $a = 1$, $K = 1$, and $M = 1$. By L'Hôpital's Rule (twice)

$$\lim_{t\to\infty} \frac{t^2}{e^t} = \lim_{t\to\infty} \frac{2}{e^t} = 0.$$

- $f(t) = e^{t^2}$ is **NOT** of **exponential order**

Existence of Laplace Transform

Theorem (Existence of Laplace Transform)

Suppose

1. f *is* ***piecewise continuous*** *on the interval* $0 \leq t \leq A$ *for any positive* A
2. f *is of* ***exponential order****, i.e., there exist real constants* $M \geq 0$, $K > 0$, *and* a, *such that*

$$|f(t)| \leq Ke^{at},$$

when $t \geq M$.

Then the ***Laplace transform*** *given by*

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt,$$

exists for $s > a$.

Short Table of Laplace Transforms

Short Table of Laplace Transforms: Below is a short table of Laplace transforms for some elementary functions

$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad$ integer $n > 0$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$

Laplace Transform - $e^{ct}f(t)$

Laplace Transform - $e^{ct}f(t)$: Previously found **Laplace transforms** of several basic functions

Theorem (Exponential Shift Theorem)

If $F(s) = \mathcal{L}[f(t)]$ *exists for* $s > a$, *and if* c *is a constant, then*

$$\mathcal{L}[e^{ct}f(t)] = F(s-c), \qquad s > a + c.$$

Proof:
This result immediately follows from the definition:

$$\mathcal{L}[e^{ct}f(t)] = \int_0^\infty e^{-st}e^{ct}f(t)dt = \int_0^\infty e^{-(s-c)t}f(t)dt = F(s-c),$$

which holds for $s - c > a$.

Example

Example: Consider the function

$$g(t) = e^{-2t}\cos(3t).$$

From our **Table of Laplace Transforms**, if $f(t) = \cos(3t)$, then

$$F(s) = \frac{s}{s^2+9}, \qquad s > 0.$$

From our previous theorem, the **Laplace transform** of $g(t)$ satisfies:

$$G(s) = \mathcal{L}[e^{-2t}f(t)] = F(s+2) = \frac{s+2}{(s+2)^2+9}, \qquad s > -2.$$

Laplace Transform of Derivatives 1

Theorem (Laplace Transform of Derivatives)

Suppose that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Suppose that f and f' are of exponential order with $|f^{(i)}(t)| \le Ke^{at}|$ for some constants K and a and $i = 0, 1$. Then $\mathcal{L}[f'(t)]$ exists for $s > a$, and moreover

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0).$$

Sketch of Proof: If $f'(t)$ was continuous, then examine

$$\begin{aligned} \int_0^A e^{-st} f'(t)dt &= e^{-st}f(t)\Big|_0^A + s\int_0^A e^{-st}f(t)dt \\ &= e^{-sA}f(A) - f(0) + s\int_0^A e^{-st}f(t)dt, \end{aligned}$$

which simply uses integration by parts.

Laplace Transform of Derivatives 2

Sketch of Proof (cont): From before we have

$$\int_0^A e^{-st} f'(t)dt = e^{-sA} f(A) - f(0) + s\int_0^A e^{-st} f(t)dt.$$

As $A \to \infty$ and using the exponential order of f and f', this expression gives

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0).$$

To complete the general proof with $f'(t)$ being piecewise continuous, we divide the integral into subintervals where $f'(t)$ is continuous.

Each of these integrals is integrated by parts, then continuity of $f(t)$ collapses the end point evaluations and allows the single integral noted on the right hand side, completing the general proof.

Laplace Transform of Derivatives 3

Corollary (Laplace Transform of Derivatives)

Suppose that

1. *The functions* f, f', f'', ..., $f^{(n-1)}$ *are continuous and that* $f^{(n)}$ *is piecewise continuous on any interval* $0 \le t \le A$
2. *The functions* f, f', ..., $f^{(n)}$ *are of exponential order with* $|f^{(i)}(t)| \le Ke^{at}$ *for some constants* K *and* a *and* $0 \le i \le n$.

Then $\mathcal{L}[f^{(n)}(t)]$ *exists for* $s > a$ *and satisfies*

$$\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

For our 2^{nd} order differential equations we will commonly use

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0).$$

Laplace Transform of Derivatives - Example

Example: Consider

$$g(t) = e^{-2t}\sin(4t) \qquad \text{with} \qquad g'(t) = -2e^{-2t}\sin(4t) + 4e^{-2t}\cos(4t)$$

If $f(t) = \sin(4t)$, then

$$F(s) = \frac{4}{s^2+16}, \qquad \text{with} \qquad G(s) = \frac{4}{(s+2)^2+16}$$

using the exponential theorem of Laplace transforms

Our derivative theorem gives

$$\mathcal{L}[g'(t)] = sG(s) - g(0) = \frac{4s}{(s+2)^2+16}$$

However,

$$\begin{aligned} \mathcal{L}[g'(t)] &= -2\mathcal{L}[e^{-2t}\sin(4t)] + 4\mathcal{L}[e^{-2t}\cos(4t)] \\ &= \frac{-8}{(s+2)^2+16} + \frac{4(s+2)}{(s+2)^2+16} = \frac{4s}{(s+2)^2+16} \end{aligned}$$

Laplace Transform of Derivatives - Example

Example: Consider the initial value problem:

$$y'' + 2y' + 5y = e^{-t}, \qquad y(0) = 1, \qquad y'(0) = -3$$

Taking **Laplace Transforms** we have

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] + 5\mathcal{L}[y] = \mathcal{L}[e^{-t}]$$

With $Y(s) = \mathcal{L}[y(t)]$, our derivative theorems give

$$s^2Y(s) - sy(0) - y'(0) + 2\left[sY(s) - y(0)\right] + 5Y(s) = \frac{1}{s+1}$$

or

$$(s^2 + 2s + 5)Y(s) = \frac{1}{s+1} + s - 1$$

We can write

$$Y(s) = \frac{1}{(s+1)(s^2+2s+5)} + \frac{s-1}{s^2+2s+5} = \frac{s^2}{(s+1)(s^2+2s+5)}$$

Laplace Transform of Derivatives - Example

Example (cont): From before,

$$Y(s) = \frac{s^2}{(s+1)(s^2+2s+5)}$$

An important result of the **Fundamental Theorem of Algebra** is **Partial Fractions Decomposition**

We write

$$Y(s) = \frac{s^2}{(s+1)(s^2+2s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+5}$$

Equivalently,

$$s^2 = A(s^2+2s+5) + (Bs+C)(s+1)$$

Let $s = -1$, then $1 = 4A$ or $A = \frac{1}{4}$

Coefficient of s^2 gives $1 = A + B$, so $B = \frac{3}{4}$

Coefficient of s^0 gives $0 = 5A + C$, so $C = -\frac{5}{4}$

Laplace Transform of Derivatives - Example

Example (cont): From the **Partial Fractions Decomposition** with $A = \frac{1}{4}$, $B = \frac{3}{4}$, and $C = -\frac{5}{4}$,

$$Y(s) = \frac{1}{4}\left(\frac{1}{s+1} + \frac{3s-5}{s^2+2s+5}\right) = \frac{1}{4}\left(\frac{1}{s+1} + \frac{3(s+1)-8}{(s+1)^2+4}\right)$$

Equivalently, we can write this

$$Y(s) = \frac{1}{4}\left(\frac{1}{s+1} + 3\frac{(s+1)}{(s+1)^2+4} - 4\frac{2}{(s+1)^2+4}\right)$$

However, $\mathcal{L}[e^{-t}] = \frac{1}{s+1}$, $\mathcal{L}[e^{-t}\cos(2t)] = \frac{s+1}{(s+1)^2+4}$, and $\mathcal{L}[e^{-t}\sin(2t)] = \frac{2}{(s+1)^2+4}$, so inverting the **Laplace transform** gives

$$y(t) = \frac{1}{4}e^{-t} + \frac{3}{4}e^{-t}\cos(2t) - e^{-t}\sin(2t),$$

solving the initial value problem

More Laplace Transforms

Theorem

Suppose that f is (i) piecewise continuous on any interval $0 \leq t \leq A$, and (ii) has exponential order with exponent a. Then for any positive integer

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s), \qquad s > a.$$

Proof:

$$\begin{aligned} F^{(n)}(s) &= \frac{d^n}{ds^n}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{\partial^n}{\partial s^n}(e^{-st}) f(t)dt \\ &= \int_0^\infty (-t)^n e^{-st} f(t)dt = (-1)^n \int_0^\infty t^n e^{-st} f(t)dt \\ &= (-1)^n \mathcal{L}[t^n f(t)] \end{aligned}$$

Corollary: For any integer, $n \geq 0$,

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \qquad s > 0.$$