

Math 337 - Elementary Differential Equations

Lecture Notes – Existence and Uniqueness

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Introduction

Introduction

- **Linear Differential Equation** - Unique solution easily found
- **Nonlinear Differential Equation** - Solutions difficult or impossible
 - When does a solution **exist**?
 - If there is a solution, then is it **unique**?
 - Proving there is a unique solution does not mean the solution can be found

Linear Differential Equation

Theorem

If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing a point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in I with the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

Linear Differential Equation

The **Linear Differential Equation** has a unique solution to

$$y' + p(t)y = g(t), \quad \text{with} \quad y(t_0) = y_0$$

- Assume p and g are continuous on an open interval $I : \alpha < t < \beta$
- It follows that p and g are integrable
- Obtain integrating factor

$$\mu(t) = e^{\int_{t_0}^t p(s)ds}$$

- **General solution** (previously found)

$$y(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^t \mu(s)g(s)ds + C \right)$$

- With initial condition, $C = y_0$, so unique solution

$$y(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^t \mu(s)g(s)ds + y_0 \right)$$

Nonlinear Differential Equation

The general 1st **Order Differential Equation** with an initial condition is given by

$$y' = f(t, y), \quad \text{with} \quad y(t_0) = y_0$$

- Need special conditions on $f(t, y)$ to find a solution
 - Can use **separable** technique if $f(t, y) = M(t)N(y)$
 - Many specialized methods, like **Exact** or **Bernoulli's equation**
- What conditions are needed on $f(t, y)$ for existence of a unique solution?
- With no general solution we need an indirect approach
- Technique uses convergence of a sequence of functions with methods from advanced calculus

Existence and Uniqueness

A change of coordinates allows us to consider

$$y' = f(t, y), \quad \text{with} \quad y(0) = 0 \quad (1)$$

Theorem

If f and $\partial f/\partial y$ are continuous in a rectangle $R : |t| \leq a, |y| \leq b$, then there is some interval $|t| \leq h \leq |a|$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem (1).

Motivation: Suppose that there is a function $y = \phi(t)$ that satisfies (1). Integrating, $\phi(t)$ must satisfy

$$\phi(t) = \int_{t_0}^t f(s, \phi(s)) ds, \quad (2)$$

which is an **integral equation**.

A solution to (1) is equivalent (2).

Picard Iteration

1

Show a solution to the **integral equation** using the **Method of Successive Approximations** or **Picard's Iteration Method**

Start with an initial function, $\phi_0 = 0$ (satisfying initial condition)

$$\phi_1(t) = \int_0^t f(s, \phi_0(s)) ds$$

Successively obtain

$$\begin{aligned}\phi_2(t) &= \int_0^t f(s, \phi_1(s)) ds \\ &\vdots \\ \phi_{n+1}(t) &= \int_0^t f(s, \phi_n(s)) ds\end{aligned}$$

Picard Iteration

The **Picard's Iteration** generates a sequence, so to prove the theorem we must demonstrate

- 1 Do all members of the sequence exist?
- 2 Does the sequence converge?
- 3 What are the properties of the limit function?
Does it satisfy the **integral equation**
- 4 Is this the only solution? (**Uniqueness**)

Picard Iteration - Example

1

Consider the initial value problem (IVP)

$$y' = 2t(1 + y), \quad \text{with} \quad y(0) = 0,$$

and apply the **Method of Successive Approximations**

Let $\phi_0 = 0$, then

$$\phi_1(t) = \int_0^t 2s(1 + \phi_0(s))ds = t^2$$

Next

$$\phi_2(t) = \int_0^t 2s(1 + \phi_1(s))ds = \int_0^t 2s(1 + s^2)ds = t^2 + \frac{t^4}{2}$$

Next

$$\phi_3(t) = \int_0^t 2s(1 + \phi_2(s))ds = t^2 + \frac{t^4}{2} + \frac{t^6}{2 \cdot 3}$$

Picard Iteration - Example

The integrations above suggest

$$\phi_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \dots + \frac{t^{2n}}{n!},$$

By math induction, assume true for $n = k$

$$\begin{aligned}\phi_{k+1}(t) &= \int_0^t 2s(1 + \phi_k(s))ds \\ &= \int_0^t 2s\left(1 + s^2 + \dots + \frac{s^{2k}}{k!}\right)ds \\ &= t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \dots + \frac{t^{2k+2}}{(k+1)!}\end{aligned}$$

which is what we needed to show

The limit exists if the series converges or $\lim_{n \rightarrow \infty} \phi_n(t)$ exists

Picard Iteration - Example

Apply the **Ratio test**

$$\lim_{k \rightarrow \infty} \left| \frac{t^{2k+2}}{(k+1)!} \frac{k!}{t^{2k}} \right| = \frac{t^2}{k+1} \rightarrow 0$$

which shows this series converges for all t

Since this is a Taylor's series, it can be integrated and differentiated in its interval of convergence.

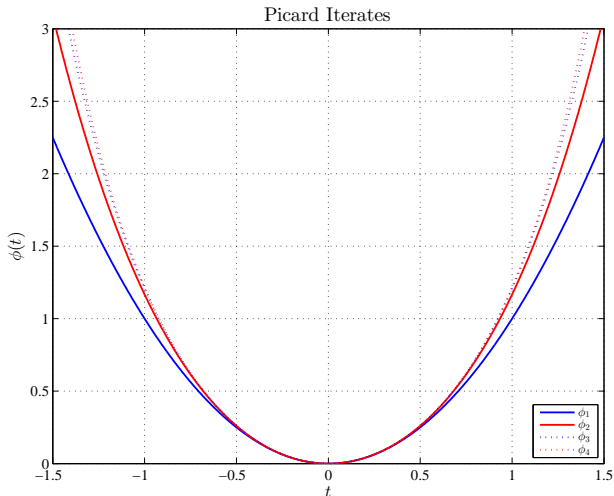
Thus, it is a solution of the **integral equation**

Note that this is the Taylor's series for $\phi(t) = e^{t^2} - 1$, which can be shown to satisfy the IVP

Picard Iteration - Example

4

First 4 Picard Iterates



Example - Uniqueness

1

Example - Uniqueness - Suppose there are two solutions, $\phi(t)$ and $\psi(t)$ satisfying the **integral equation**

$$\phi(t) - \psi(t) = \int_0^t 2s(\phi(s) - \psi(s))ds$$

Take absolute values and restrict $0 \leq t \leq A/2$ (A arbitrary). then

$$\begin{aligned} |\phi(t) - \psi(t)| &= \left| \int_0^t 2s(\phi(s) - \psi(s))ds \right| \leq \int_0^t 2s|\phi(s) - \psi(s)|ds \\ &\leq A \int_0^t |\phi(s) - \psi(s)|ds \quad \text{for } 0 \leq t \leq A/2 \end{aligned}$$

Example - Uniqueness

2

Let $U(t) = \int_0^t |\phi(s) - \psi(s)| ds$, then $U(0) = 0$ and $U(t) \geq 0$ for $t \geq 0$

$U(t)$ is differentiable with $U'(t) = |\phi(t) - \psi(t)|$

We have the differential inequality

$$U'(t) - AU(t) \leq 0, \quad 0 \leq t \leq A/2$$

Multiplying by positive function e^{-At} , then integrating gives

$$\begin{aligned} \frac{d}{dt} (e^{-At}U(t)) &\leq 0, & 0 \leq t \leq A/2, \\ e^{-At}U(t) &\leq 0, & 0 \leq t \leq A/2 \end{aligned}$$

Hence, $U(t) \leq 0$ with A arbitrary.

It follows that $U(t) \equiv 0$ or $\phi(t) = \psi(t)$ for each t , so the functions are the same, giving **uniqueness**

Existence and Uniqueness Theorem

We leave the details of the proof of the **Existence and Uniqueness Theorem** to the interested reader, but give a sketch of the key steps

- 1 Restrict the time interval $|t| \leq h \leq a$
 - Since f is continuous in the the rectangle $R : |t| \leq a, |y| \leq b$, the function f is bounded on R , so there exists M such that

$$|f(t, y)| \leq M \quad (t, y) \in R$$

- Let $h = \min\left(a, \frac{b}{M}\right)$
- Can show by induction that each Picard iterate $\phi_n(t)$ satisfies

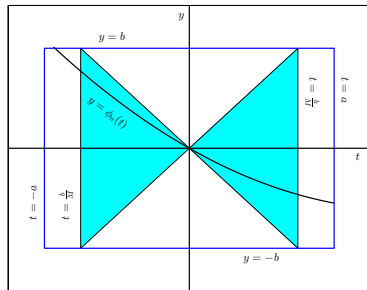
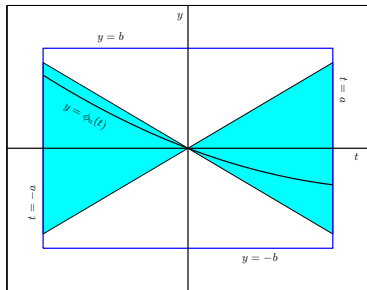
$$|\phi_n(t)| \leq Mt \quad t \in [0, h]$$

- This gives **existence** of the Picard iterates

Existence and Uniqueness Theorem

2

Sketch of Proof of Existence and Uniqueness Theorem



Regions containing Picard iterates, $\phi_n(t)$ for all n

Existence and Uniqueness Theorem

Sketch of Proof of Existence and Uniqueness Theorem

2 Show the sequence converges

- A key point in the theorem is the continuity of $\partial f/\partial y$
- Let

$$L = \max_{t \in R} \left| \frac{\partial f(t, y)}{\partial y} \right|,$$

which is called a **Lipschitz** constant

- Create a **Cauchy sequence** and show

$$|\phi_n(t) - \phi_{n-1}(t)| \leq \frac{ML^{n-1}t^n}{n!} \quad t \in [0, h]$$

- This establishes **convergence** of the Picard iterates

Existence and Uniqueness Theorem

Sketch of Proof of Existence and Uniqueness Theorem

- 8 Show the convergent sequence converges to the solution of the IVP

- The iteration scheme is

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s)) ds$$

- Want to take the limit of both sides as $n \rightarrow \infty$
- We have

$$\lim_{n \rightarrow \infty} \phi_{n+1}(t) = \phi(t) = \lim_{n \rightarrow \infty} \int_0^t f(s, \phi_n(s)) ds$$

- **Uniform convergence** of the Picard iterates allows

$$\phi(t) = \int_0^t \lim_{n \rightarrow \infty} f(s, \phi_n(s)) ds$$

Existence and Uniqueness Theorem

5

Sketch of Proof of Existence and Uniqueness Theorem

⑧ (cont) Show the convergent sequence converges to the solution of the IVP

- Continuity of $f(t, y)$ w.r.t. y allows

$$\phi(t) = \int_0^t f(s, \lim_{n \rightarrow \infty} \phi_n(s)) ds$$

- This gives convergence to the solution
- ④ Proof **Uniqueness** by producing a contradiction assuming two solutions

This proves when solutions **exist** and are **unique** to an **Initial Value Problem**

Examples

1

The general differential equation is

$$y' = f(t, y), \quad \text{with} \quad y(t_0) = y_0 \quad (3)$$

Theorem

If f and $\partial f/\partial y$ are continuous in a rectangle $R : |t - t_0| \leq a, |y - y_0| \leq b$, then there is some interval $|t - t_0| \leq h \leq |a|$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem (3).

- 1 Why do we need the restriction $|t - t_0| \leq h \leq |a|$?
- 2 What is the significance of the conditions f and $\partial f/\partial y$ being continuous in R ?

Examples

Consider the differential equation

$$y' = y^2, \quad \text{with} \quad y(0) = 1$$

Note that $f(y) = y^2$ and $\partial f/\partial y = 2y$, which are continuous in any rectangle R

This is a **separable** equation, so

$$\int y^{-2} dy = \int dt = t + C \quad \text{or} \quad -\frac{1}{y(t)} = t + C$$

The solution to the IVP is

$$y(t) = \frac{1}{1-t},$$

which clearly becomes undefined at $t = 1$. The **interval of existence** does not match the interval of continuity for $f(t, y)$

Examples

Consider the differential equation

$$y' = y^{2/3}, \quad \text{with} \quad y(0) = 0$$

Note that $f(y) = y^{2/3}$ is continuous in any rectangle centered at $(t_0, y_0) = (0, 0)$, while $\partial f / \partial y = \frac{2}{3}y^{-1/3}$, which is **NOT continuous** in any rectangle R near $(0, 0)$

This is a **separable** equation, so

$$\int y^{-2/3} dy = \int dt = t + C \quad \text{or} \quad 3y(t)^{1/3} = t + C$$

One solution to the IVP is

$$y(t) = \frac{t^3}{27},$$

which satisfies the IVP.

However, it is easy to see that $y(t) \equiv 0$ is a solution, so solutions are **NOT unique**