Math 337 - Elementary Differential Equations

Lecture Notes – Existence and Uniqueness

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Outline

Introduction

Introduction

- **Linear Differential Equation** - Unique solution easily found
- **Nonlinear Differential Equation** - Solutions difficult or impossible
  - When does a solution **exist**?
  - If there is a solution, then is it **unique**?
  - Proving there is a unique solution does not mean the solution can be found

Linear Differential Equation

Theorem

If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing a point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in I with the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

Linear Differential Equation

The **Linear Differential Equation** has a unique solution to

$$y' + p(t)y = g(t), \qquad \text{with} \qquad y(t_0) = y_0$$

- Assume p and g are continuous on an open interval $I : \alpha < t < \beta$
- It follows that p and g are integrable
- Obtain integrating factor

$$\mu(t) = e^{\int_{t_0}^{t} p(s)ds}$$

- **General solution** (previously found)

$$y(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)ds + C\right)$$

- With initial condition, $C = y_0$, so unique solution

$$y(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)ds + y_0\right)$$

Nonlinear Differential Equation

The general 1^{st} **Order Differential Equation** with an initial condition is given by

$$y' = f(t,y), \qquad \text{with} \qquad y(t_0) = y_0$$

- Need special conditions on $f(t,y)$ to find a solution
  - Can use **separable** technique if $f(t,y) = M(t)N(y)$
  - Many specialized methods, like **Exact** or **Bernoulli's equation**
- What conditions are needed on $f(t,y)$ for existence of a unique solution?
- With no general solution we need an indirect approach
- Technique uses convergence of a sequence of functions with methods from advanced calculus

Existence and Uniqueness

A change of coordinates allows us to consider

$$y' = f(t,y), \qquad \text{with} \qquad y(0) = 0 \tag{1}$$

Theorem

If f and $\partial f/\partial y$ are continuous in a rectangle $R : |t| \le a, |y| \le b$, then there is some interval $|t| \le h \le |a|$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem (1).

Motivation: Suppose that there is a function $y = \phi(t)$ that satisfies (1). Integrating, $\phi(t)$ must satisfy

$$\phi(t) = \int_{t_0}^{t} f(s, \phi(s))ds, \tag{2}$$

which is an integral equation.

A solution to (1) is equivalent (2).

Picard Iteration 1

Show a solution to the integral equation using the **Method of Successive Approximations** or **Picard's Iteration Method**

Start with an initial function, $\phi_0 = 0$ (satisfying initial condition)

$$\phi_1(t) = \int_0^t f(s, \phi_0(s))\, ds$$

Successively obtain

$$\begin{aligned} \phi_2(t) &= \int_0^t f(s, \phi_1(s))\, ds \\ &\vdots \\ \phi_{n+1}(t) &= \int_0^t f(s, \phi_n(s))\, ds \end{aligned}$$

Picard Iteration 2

The **Picard's Iteration** generates a sequence, so to prove the theorem we must demonstrate

1. Do all members of the sequence exist?
2. Does the sequence converge?
3. What are the properties of the limit function?
   Does it satisfy the **integral equation**
4. Is this the only solution? (**Uniqueness**)

Picard Iteration - Example 1

Consider the initial value problem (IVP)

$$y' = 2t(1+y), \qquad \text{with} \qquad y(0) = 0,$$

and apply the **Method of Successive Approximations**

Let $\phi_0 = 0$, then

$$\phi_1(t) = \int_0^t 2s(1+\phi_0(s))ds = t^2$$

Next

$$\phi_2(t) = \int_0^t 2s(1+\phi_1(s))ds = \int_0^t 2s(1+s^2)ds = t^2 + \frac{t^4}{2}$$

Next

$$\phi_3(t) = \int_0^t 2s(1+\phi_2(s))ds = t^2 + \frac{t^4}{2} + \frac{t^6}{2\cdot 3}$$

Picard Iteration - Example 2

The integrations above suggest

$$\phi_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + ... + \frac{t^{2n}}{n!},$$

By math induction, assume true for $n = k$

$$\begin{aligned}\phi_{k+1}(t) &= \int_0^t 2s(1+\phi_k(s))ds \\ &= \int_0^t 2s(1+s^2+...+\frac{s^{2k}}{k!})ds \\ &= t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + ... + \frac{t^{2k+2}}{(k+1)!}\end{aligned}$$

which is what we needed to show

The limit exists if the series converges or $\lim_{n\to\infty}\phi_n(t)$ exists

Picard Iteration - Example 3

Apply the **Ratio test**

$$\lim_{k\to\infty}\left|\frac{t^{2k+2}}{(k+1)!}\frac{k!}{t^{2k}}\right| = \frac{t^2}{k+1} \to 0$$

which shows this series converges for all t

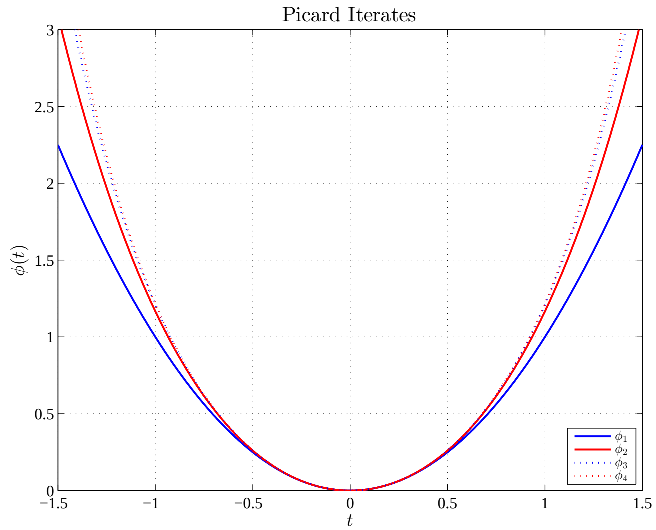
Since this is a Taylor's series, it can be integrated and differentiated in its interval of convergence.

Thus, it is a solution of the **integral equation**

Note that this is the Taylor's series for $\phi(t) = e^{t^2} - 1$, which can be shown to satisfy the IVP

Picard Iteration - Example 4

First 4 Picard Iterates



Example - Uniqueness 1

Example - Uniqueness - Suppose there are two solutions, $\phi(t)$ and $\psi(t)$ satisfying the **integral equation**

$$\phi(t) - \psi(t) = \int_0^t 2s(\phi(s) - \psi(s))ds$$

Take absolute values and restrict $0 \le t \le A/2$ (A arbitrary). then

$$\begin{aligned} |\phi(t) - \psi(t)| &= \left|\int_0^t 2s(\phi(s) - \psi(s))ds\right| \le \int_0^t 2s|\phi(s) - \psi(s)|ds \\ &\le A\int_0^t |\phi(s) - \psi(s)|ds \qquad \text{for} \qquad 0 \le t \le A/2 \end{aligned}$$

Example - Uniqueness 2

Let $U(t) = \int_0^t |\phi(s) - \psi(s)|ds$, then $U(0) = 0$ and $U(t) \ge 0$ for $t \ge 0$

$U(t)$ is differentiable with $U'(t) = |\phi(t) - \psi(t)|$

We have the differential inequality

$$U'(t) - AU(t) \le 0, \qquad 0 \le t \le A/2$$

Multiplying by positive function e^{-At}, then integrating gives

$$\begin{aligned} \frac{d}{dt}\left(e^{-At}U(t)\right) &\le 0, \qquad 0 \le t \le A/2, \\ e^{-At}U(t) &\le 0, \qquad 0 \le t \le A/2 \end{aligned}$$

Hence, $U(t) \le 0$ with A arbitrary.

It follows that $U(t) \equiv 0$ or $\phi(t) = \psi(t)$ for each t, so the functions are the same, giving **uniqueness**

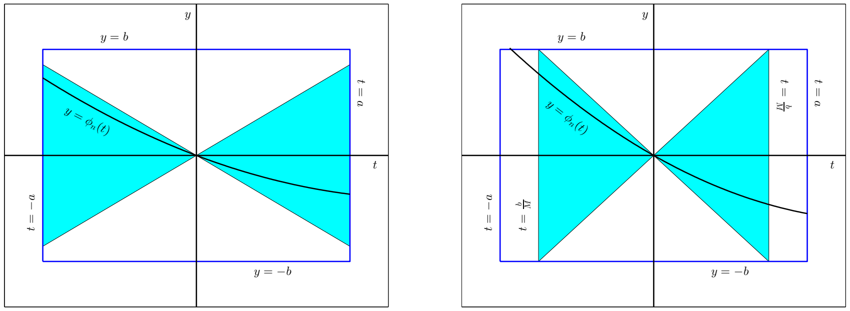
Existence and Uniqueness Theorem 1

We leave the details of the proof of the **Existence and Uniqueness Theorem** to the interested reader, but give a sketch of the key steps

1. Restrict the time interval $|t| \le h \le a$
   - Since f is continuous in the the rectangle $R : |t| \le a, |y| \le b$, the function f is bounded on R, so there exists M such that
   $$|f(t,y)| \le M \qquad (t,y) \in R$$
   - Let $h = \min\left(a, \frac{b}{M}\right)$
   - Can show by induction that each Picard iterate $\phi_n(t)$ satisfies
   $$|\phi_n(t)| \le Mt \qquad t \in [0,h]$$
   - This gives **existence** of the Picard iterates

Existence and Uniqueness Theorem 2

Sketch of Proof of Existence and Uniqueness Theorem



Regions containing Picard iterates, $\phi_n(t)$ for all n

Existence and Uniqueness Theorem 3

Sketch of Proof of Existence and Uniqueness Theorem

2. Show the sequence converges
   - A key point in the theorem is the continuity of $\partial f/\partial y$
   - Let
   $$L = \max_{t \in R} \left| \frac{\partial f(t,y)}{\partial y} \right|,$$
   which is called a **Lipschitz** constant
   - Create a **Cauchy sequence** and show
   $$|\phi_n(t) - \phi_{n-1}(t)| \leq \frac{ML^{n-1}t^n}{n!} \qquad t \in [0, h]$$
   - This establishes **convergence** of the Picard iterates

Existence and Uniqueness Theorem 4

Sketch of Proof of Existence and Uniqueness Theorem

3. Show the convergent sequence converges to the solution of the IVP
   - The iteration scheme is
   $$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$$
   - Want to take the limit of both sides as $n \to \infty$
   - We have
   $$\lim_{n\to\infty} \phi_{n+1}(t) = \phi(t) = \lim_{n\to\infty} \int_0^t f(s, \phi_n(s))ds$$
   - **Uniform convergence** of the Picard iterates allows
   $$\phi(t) = \int_0^t \lim_{n\to\infty} f(s, \phi_n(s))ds$$

Existence and Uniqueness Theorem 5

Sketch of Proof of Existence and Uniqueness Theorem

3. (cont) Show the convergent sequence converges to the solution of the IVP
   - Continuity of $f(t, y)$ w.r.t. y allows
   $$\phi(t) = \int_0^t f(s, \lim_{n\to\infty} \phi_n(s))ds$$
   - This gives convergence to the solution
4. Proof **Uniqueness** by producing a contradiction assuming two solutions

This proves when solutions **exist** and are **unique** to an **Initial Value Problem**

Examples 1

The general differential equation is

$$y' = f(t,y), \qquad \text{with} \qquad y(t_0) = y_0 \tag{3}$$

Theorem

If f and $\partial f/\partial y$ are continuous in a rectangle $R: |t-t_0| \le a, |y-y_0| \le b$, then there is some interval $|t-t_0| \le h \le |a|$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem (3).

1. Why do we need the restriction $|t-t_0| \le h \le |a|$?
2. What is the significance of the conditions f and $\partial f/\partial y$ being continuous in R?

Examples 2

Consider the differential equation

$$y' = y^2, \qquad \text{with} \qquad y(0) = 1$$

Note that $f(y) = y^2$ and $\partial f/\partial y = 2y$, which are continuous in any rectangle R

This is a **separable** equation, so

$$\int y^{-2} dy = \int dt = t + C \qquad \text{or} \qquad -\frac{1}{y(t)} = t + C$$

The solution to the IVP is

$$y(t) = \frac{1}{1-t},$$

which clearly becomes undefined at $t = 1$. The **interval of existence** does not match the interval of continuity for $f(t,y)$

Examples 3

Consider the differential equation

$$y' = y^{2/3}, \qquad \text{with} \qquad y(0) = 0$$

Note that $f(y) = y^{2/3}$ is continuous in any rectangle centered at $(t_0, y_0) = (0,0)$, while $\partial f/\partial y = \frac{2}{3}y^{-1/3}$, which is **NOT continuous** in any rectangle R near $(0,0)$

This is a **separable** equation, so

$$\int y^{-2/3} dy = \int dt = t + C \qquad \text{or} \qquad 3y(t)^{1/3} = t + C$$

One solution to the IVP is

$$y(t) = \frac{t^3}{27},$$

which satisfies the IVP.

However, it is easy to see that $y(t) \equiv 0$ is a solution, so solutions are **NOT unique**