Calculus for the Life Sciences

Lecture Notes – Optimization

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Outline

Introduction

Introduction

- Animals are frequently devising **optimal strategies** to gain advantage
  - Reproducing more rapidly
  - Better protection from predation
- Primitive animals long ago split into the **prokaryotes** (bacterial cells) and **eukaryotes** (cells in higher organisms like yeast or humans) from a common ancestor
  - One argument contends that eukaryotic cells added complexity, size, and organization for advantage in competition
  - Prokaryotes stripped down their genome (eliminated junk DNA) to the minimum required for survival, to maximize reproduction
- These arguments suggest that organisms try to optimize their situation to gain an advantage

Crow Predation on Whelks

Crow Predation on Whelks

- Sea gulls and crows have learned to feed on various mollusks by dropping their prey on rocks to break the protective shells

Optimal Foraging

Optimal Foraging – Northwestern crows (*Corvus caurinus*) on Mandarte Island

- Reto Zach studied Northwestern crows on Mandarte Island, British Columbia to learn about foraging for whelks (*Thais lamellosa*)
- Ecologists study these behaviors to give insight into **optimal foraging**
- Northwestern crows
  - Perch above beaches, then fly to intertidal zone
  - Select largest whelks
  - Fly to the rocky area and drop whelks
  - Eat broken whelks

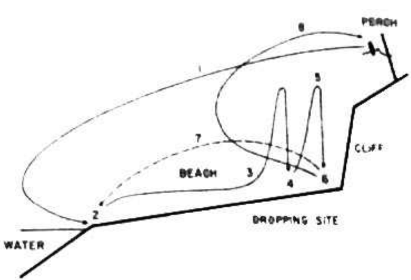
Foraging Strategy

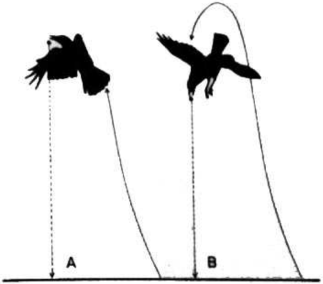
Foraging Strategy

- Whelk Selection
  - Crows search intertidal zone for largest whelks
  - Take whelks to a favorite rocky area
- Flight Strategy
  - Fly to height of about 5 meters
  - Drop whelks on rocks, repeatedly averaging 4 times
  - Eat edible parts when split open
- **Can this behavior be explained by an optimal foraging decision process?**
- **Is the crow exhibiting a behavior that minimizes its expenditure of energy to feed on whelks?**

Foraging Strategy

Foraging Strategy

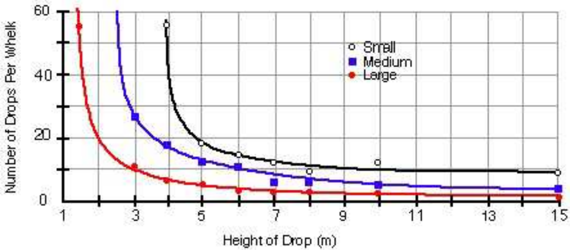




Why large whelks?

Why large whelks?

- Zach experiment
  - Collected and sorted whelks by size
  - Dropped whelks from various heights until they broke
  - Recorded how many drops at each height were required to break each whelk



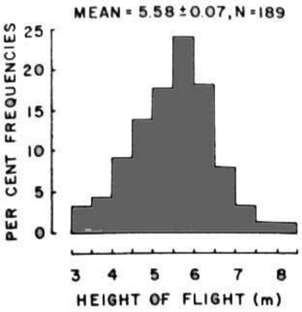
Large Whelks

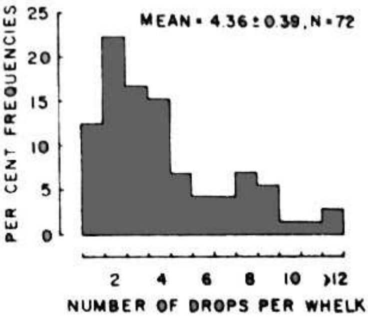
Large Whelks

- Easier to break open larger whelks, so crows selectively chose the largest available whelks
- There was a gradient of whelk size on the beach, suggesting that the crows' foraging behavior was affecting the distribution of whelks in the intertidal zone, with larger whelks further out
- Crows benefit by selecting the larger ones because they don't need as many drops per whelk, and they gain more energy from consuming a larger one
- Study showed that the whelks broken on the rocks were remarkably similar in size, weighing about 9 grams

Number of Drops

Zach Observation – Height of the drops and number of drops required for many crows to eat whelks used a marked pole on the beach near a favorite dropping location





Optimization Problem

Optimization Problem

- So why do the crows consistently fly to about 5.25 m and use about 4.4 drops to split open a whelk?
- Can this be explained by a mathematical model for minimizing the energy spent, thus supporting an optimal foraging strategy?

Mathematical Model for Energy

- Energy is directly proportional to the vertical height that an object is lifted (Work put into a system)
- The energy that a crow expends breaking open a whelk
  - The amount of time the crow uses to search for an appropriate whelk
  - The energy in flying to the site where the rocks are
  - The energy required to lift the whelk to a certain height and drop it times the number of vertical flights required to split open the whelk
- Concentrate only on this last component of the problem, as it was observed that the crows kept with the same whelk until they broke it open rather than searching for another whelk when one failed to break after a few attempts

Mathematical Model for Energy 2

Energy Function

- The energy is given by the height (H) times the number of drops (N) or
$$E = kHN$$
where k is a constant of proportionality
- Flying higher and increasing the number of drops both increase the use of energy

Mathematical Model for Energy 3

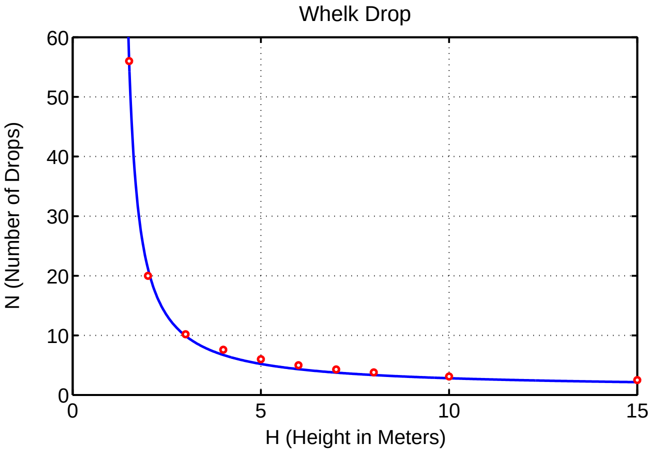
Fitting the Data – Zach's data on dropping large whelks

$H(m)$	1.5	2	3	4	5	6	7	8	10	15
$N(H)$	56	20	10.2	7.6	6	5	4.3	3.8	3.1	2.5

- Since it always requires at least one drop, the proposed function for the number of drops, N, as a function of height, H is
$$N(H) = 1 + \frac{a}{H - b}$$
- The least squares best fit of this function to Zach's data gives $a = 15.97$ and $b = 1.209$

Mathematical Model for Energy

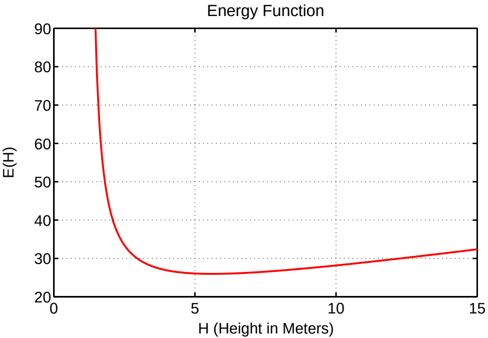
Graph for Whelks being Dropped



Mathematical Model for Energy

Graph of Energy Function – The energy function is

$$E(H) = kH\left(1 + \frac{a}{H-b}\right)$$



Mathematical Model for Energy 6

Minimization Problem – Energy satisfies

$$E(H) = kH\left(1 + \frac{a}{H-b}\right)$$

- A **minimum energy** is apparent from the graph with the value around 5.6 m, which is close to the observed value that Zach found the crows to fly when dropping whelks
- The derivative of $E(H)$ is

$$E'(H) = k\left(1 + \frac{a}{H-b} - \frac{aH}{(H-b)^2}\right) = k\left(\frac{H^2 - 2bH + b^2 - ab}{(H-b)^2}\right)$$

- The **optimal energy** occurs at the **minimum**, where

$$E'(H) = 0$$

Mathematical Model for Energy 7

Minimization Problem – The derivative of the Energy function is

$$E'(H) = k\left(\frac{H^2 - 2bH + b^2 - ab}{(H-b)^2}\right)$$

- The derivative is zero if the numerator is zero
- The numerator is a quadratic with solution

$$H = b \pm \sqrt{ab} = 1.209 \pm 4.394$$

- Thus, $H = 5.603$ is the **minimum energy** ($H = -3.185$ is a maximum, but fails to make sense)
- This computed minimum concurs with the experimental observations, suggesting an **optimal foraging strategy**

Optimal Solution

Optimal Solution

- One application of the derivative is to find critical points where often a function has a **relative minimum** or **maximum**
- An **optimal solution** for a function is when the function takes on an **absolute minimum** or **maximum** over its domain

Definition: An **absolute minimum** for a function $f(x)$ occurs at a point $x = c$, if $f(c) < f(x)$ for all x in the domain of f

Optimal Solution

Optimal Solution

Theorem: Suppose that $f(x)$ is a continuous, differential function on a closed interval $I = [a, b]$, then $f(x)$ achieves its **absolute minimum** (or maximum) on I and its minimum (or maximum) occurs either at a point where $f'(x) = 0$ or at one of the **endpoints of the interval**

Absolute Extrema of a Polynomial 1

Absolute Extrema of a Polynomial: Consider the cubic polynomial $f(x)$ defined on the interval $x \in [0, 5]$, where

$$f(x) = x^3 - 6\,x^2 + 9\,x + 4$$

Find the **absolute extrema** of this polynomial on its **domain**

Skip Example

Absolute Extrema of a Polynomial 2

Solution: The cubic polynomial

$$f(x) = x^3 - 6\,x^2 + 9\,x + 4$$

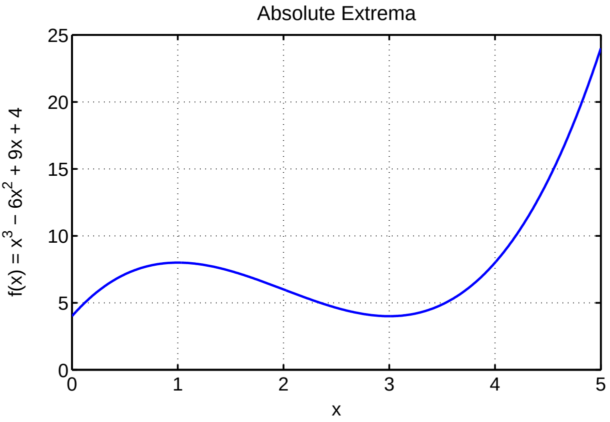
- The derivative is

$$f'(x) = 3\,x^2 - 12\,x + 9 = 3(x-1)(x-3)$$

- Critical points occur at $x_c = 1$ and $x_c = 3$
- To find the **absolute extrema**, we evaluate $f(x)$ at the **critical points** and the **endpoints** of the domain
  - $f(0) = 4$ (an **absolute minimum**)
  - $f(1) = 8$ (an **relative maximum**)
  - $f(3) = 4$ (an **absolute minimum**)
  - $f(5) = 24$ (an **absolute maximum**)

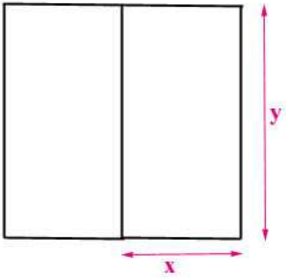
Absolute Extrema of a Polynomial 3

Solution: Graph of cubic polynomial



Optimal Study Area 1

Optimal Study Area: An ecology student goes into the field with 120 m of string and wants to create two adjacent rectangular study areas with the maximum area possible



Optimal Study Area 2

Solution – Optimal Study Area: The **Objective Function** for this problem is the area of the rectangular plots

The area of each rectangular plot is

$$A(x, y) = xy$$

The **optimal solution** uses all string

The **Constraint Condition** is the length of string available

$$P(x, y) = 4\,x + 3\,y = 120$$

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Optimal Study Area 3

Solution (cont): This problem allows the **objective function** of two variables to be reduced by the **constraint condition** to a function of one variable that can readily be **optimized**

- The **constraint condition** is solved for y to give

$$y = \frac{120 - 4\,x}{3}$$

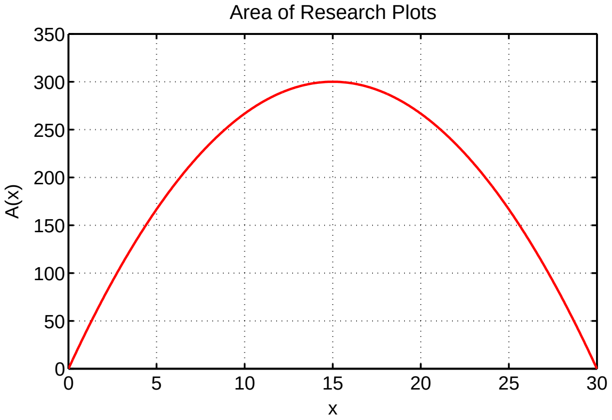
- The **objective function** becomes

$$A(x) = x\frac{120 - 4\,x}{3} = 40\,x - \frac{4\,x^2}{3}$$

- The **domain** of this function is $x \in [0, 30]$

Optimal Study Area

Solution (cont): The **objective function** is a parabola



Optimal Study Area

Solution (cont): The **optimal solution** is the **maximum area** for the function

$$A(x) = 40\,x - \frac{4\,x^2}{3}$$

- The **maximum area** occurs at the **vertex** of this parabola
- Alternately, we differentiate the **objective function** with

$$A'(x) = 40 - \frac{8\,x}{3}$$

- The **critical point** occurs when $A'(x_c) = 0$ or $x_c = 15$
- The **maximum area** occurs with $x = 15$ m and $y = 20$ m
- To maximize the study areas, the ecology student should make each of the two study areas **15 m wide** and **20 m long** or $A_{max} = \mathbf{300\ m^2}$

Chemical Reaction

Chemical Reaction: One of the simplest chemical reactions is the combination of two substances to form a third

$$A + B \xrightarrow{k} X$$

- Assume the initial concentration of substance A is a and the initial concentration of B is b
- The law of mass action gives the following reaction rate

$$R(x) = k(a - x)(b - x), \qquad 0 \le x \le \min(a, b)$$

- k is the rate constant of the reaction and x is the concentration of X during the reaction
- What is the concentration of X where the reaction rate is at a maximum?

Chemical Reaction 2

Chemical Reaction: Suppose that $k = 50$ (sec^{-1}), $a = 6$ (ppm), and $b = 2$ (ppm), so

$$R(x) = 50(6 - x)(2 - x) = 50\,x^2 - 400\,x + 600, \quad 0 \le x \le 2$$

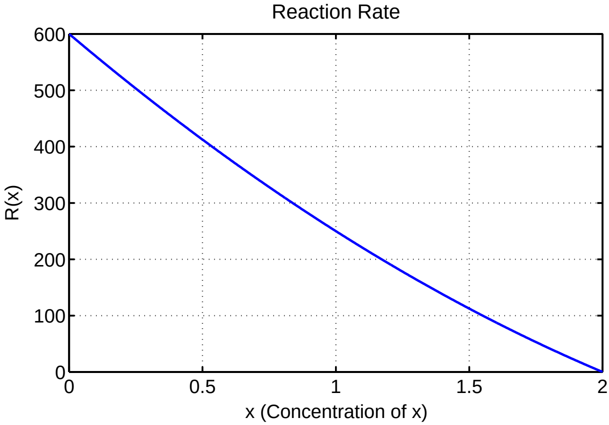
- The derivative is

$$R'(x) = 100\,x - 400$$

- The critical point (where $R'(x) = 0$) is $x_c = 4$
- This critical point is outside the domain (and produces a negative reaction rate)
- At the **endpoints**
  - At $x = 0$,the reaction rate is $R(0) = 600$ (**maximum**)
  - At $x = 2$,the reaction rate is $R(2) = 0$ (**minimum**)

Chemical Reaction

Chemical Reaction: Graphing the Reaction Rate



Nectar Foraging 1

Example - Nectar Foraging by Bumblebees[1]: Animals often forage on resources that are in discrete patches

- Bumblebees forage on many flowers
- The amount of nectar, $N(t)$, consumed increases with diminishing returns with time t
- Suppose this function satisfies

$$N(t) = \frac{0.3t}{t+2},$$

with t in sec and N in mg

- Assume the travel time between flowers is 4 sec

[1] J. Stewart and T. Day, *Biocalculus*, Cengage (2015)

Nectar Foraging 2

Nectar Foraging: Assume bumblebees acquire nectar according to

$$N(t) = \frac{0.3t}{t+2},$$

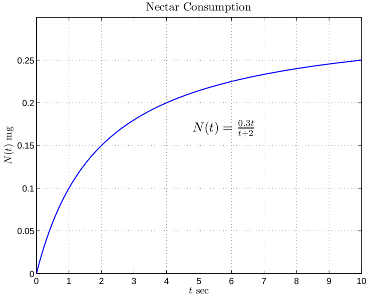
with travel between flowers being 4 sec

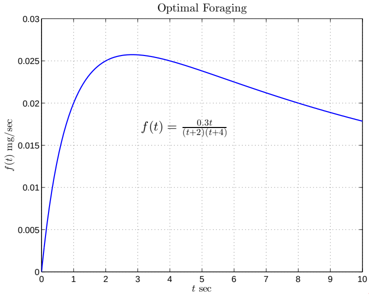
- If the bee spends t sec at each flower, then create a foraging function, $f(t)$, describing the nectar consumed over one cycle from landing on one flower until landing on the next flower
- Find the **optimal** foraging time for receiving the maximum energy gain from the nectar

Nectar Foraging

Nectar Foraging: The time for a complete foraging cycle is $t+4$, so the **nectar consumed per second** is

$$f(t) = \frac{N(t)}{t+4} = \frac{0.3t}{(t+2)(t+4)} = \frac{0.3t}{t^2+6t+8}$$





Nectar Foraging 4

Optimal Nectar Foraging: Since

$$f(t) = \frac{N(t)}{t+4} = \frac{0.3t}{(t+2)(t+4)} = \frac{0.3t}{t^2+6t+8},$$

the derivative satisfies

$$f'(t) = \frac{0.3(t^2+6t+8) - 0.3t(2t+6)}{(t^2+6t+8)^2} = \frac{-0.3t^2+2.4}{(t^2+6t+8)^2},$$

The maximum of $f(t)$ is when $f'(t) = 0$, which occurs when

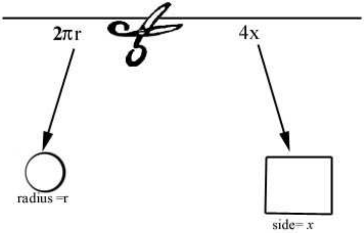
$$-0.3t^2 + 2.4 = 0 \qquad \text{or} \qquad t = 2\sqrt{2} \approx 2.83 \text{ sec}$$

Wire Problem 1

Example: Wire Problem A wire length L is cut to make a circle and a square

Skip Example

How should the cut be made to maximize the area enclosed by the two shapes?



Wire Problem 2

Solution: The circle has area πr^2, and the square has area x^2

The **Objective Function** to be **optimized** is

$$A(r,x) = \pi r^2 + x^2$$

The **Constraint Condition** based on the length of the wire

$$L = 2\pi r + 4\,x$$

with **domain** $x \in [0, \frac{L}{4}]$

From the constraint, r satisfies

$$r = \frac{L - 4\,x}{2\pi}$$

Wire Problem 3

Solution: With the constraint condition, the area function becomes

$$A(x) = \frac{(L - 4\,x)^2}{4\pi} + x^2$$

Differentiating $A(x)$ gives

$$A'(x) = \frac{2(L - 4x)(-4)}{4\pi} + 2x = 2\left(\left(\frac{4+\pi}{\pi}\right)x - \frac{L}{\pi}\right)$$

Relative extrema satisfy $A'(x) = 0$, so

$$(4 + \pi)x = L$$

Wire Problem 4

Solution: The **relative extremum** occurs at

$$x = \frac{L}{4+\pi}$$

- The second derivative of $A(x)$ is

$$A''(x) = \frac{8}{\pi} + 2 > 0$$

- The function is concave upward, so the **critical point** is a **minimum**
- $A(x)$ is a quadratic with the leading coefficient being positive, so the vertex of the parabola is the minimum
- Cutting the wire at $x = \frac{L}{4+\pi}$ gives the minimum possible area

Wire Problem

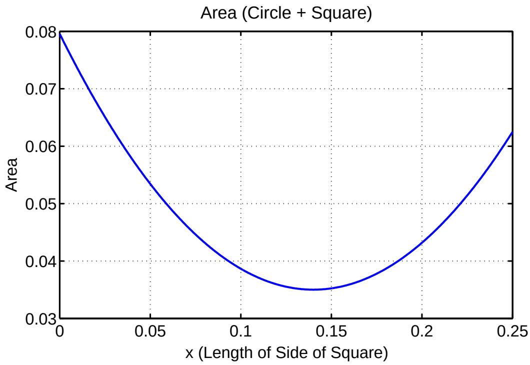
Solution: To find the **maximum** the **Theorem for an Optimal Solution** requires checking the endpoints

- The **endpoints**
  - All in the circle, $x = 0$, $A(0) = \frac{L^2}{4\pi}$
  - All in the square, $x = \frac{L}{4}$, $A\left(\frac{L}{4}\right) = \frac{L^2}{16}$
- Since $4\pi < 16$, $A(0) > A\left(\frac{L}{4}\right)$
- The **maximum** occurs when the wire is used to create a circle
- Geometrically, a circle is the most efficient conversion of a linear measurement into area

Wire Problem

Solution: Graph of **wire problem** with $L = 1$

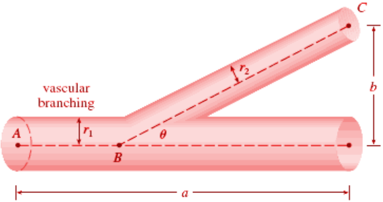
$$A(x) = \frac{(1 - 4\,x)^2}{4\pi} + x^2$$



Blood Vessel Branching 1

A smaller **blood vessel**, radius r_2, is shown branching off a primary blood vessel, radius r_1 with angle θ

Our goal is to find the **optimal angle of branching** that minimizes the energy requires to transport the blood



The primary loss of energy for flowing blood is the resistance in the blood vessels

Resistance follows **Poiseuille's Law**

$$R = C\frac{L}{r^4},$$

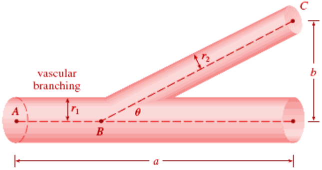
where C is a constant, L is the length of the vessel, and r is the radius of the blood vessel

Blood Vessel Branching 2

Problem begins by finding the lengths $|AB|$ and $|BC|$ as a function of θ with respect to the fixed distances a and b

From our
basic definitions of trig functions we have

$$\sin(\theta) = \frac{b}{|BC|} \quad \text{or} \quad |BC| = \frac{b}{\sin(\theta)}$$



Also,

$$\cos(\theta) = \frac{a - |AB|}{|BC|} = \frac{(a - |AB|)\sin(\theta)}{b} \quad \text{or} \quad |AB| = a - \frac{b\cos(\theta)}{\sin(\theta)}$$

Blood Vessel Branching 3

Apply Poiseuille's Law using the lengths $|AB|$ and $|BC|$ to obtain the resistance $R(\theta)$

$$R(\theta) = C\left(\frac{a}{r_1^4} - \frac{b\cos(\theta)}{r_1^4\sin(\theta)} + \frac{b}{r_2^4\sin(\theta)}\right)$$

Differentiating $R(\theta)$ gives

$$R'(\theta) = Cb\left[-\frac{1}{r_1^4}\left(\frac{-\sin(\theta)\sin(\theta) - \cos(\theta)\cos(\theta)}{\sin^2(\theta)}\right) - \frac{\cos(\theta)}{r_2^4\sin^2(\theta)}\right]$$

$$R'(\theta) = \frac{Cb}{\sin^2(\theta)}\left(\frac{1}{r_1^4} - \frac{\cos(\theta)}{r_2^4}\right)$$

Blood Vessel Branching 5

Since

$$R'(\theta) = \frac{Cb}{\sin^2(\theta)}\left(\frac{1}{r_1^4} - \frac{\cos(\theta)}{r_2^4}\right),$$

the **minimum resistance** occurs when $R'(\theta) = 0$ or

$$\cos(\theta) = \frac{r_2^4}{r_1^4} \qquad \text{or} \qquad \theta = \arccos\left(\frac{r_2^4}{r_1^4}\right)$$

For small ratios of $\frac{r_2}{r_1}$, the new blood vessel optimally comes off at nearly a right angle

When $\frac{r_2}{r_1} = \frac{1}{2}$, the optimal angle is $\theta = 1.5083 \approx 86.4°$

Even when $\frac{r_2}{r_1} = \frac{3}{4}$, the optimal angle is $\theta = 1.2489 \approx 71.6°$