

## Approximation Theory

## Trigonometric Polynomial Approximation

## Lecture Notes #14

Joe Mahaffy

Department of Mathematics

San Diego State University

San Diego, CA 92182-7720

mahaffy@math.sdsu.edu

<http://www-rohan.sdsu.edu/~jmahaffy>

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$$P(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + i \sum_{n=0}^{\infty} a_n \sin(nx)$$

- 1750s** Jean Le Rond d'Alembert used finite sums of sin and cos to study vibrations of a string.
- 17xx** Use adopted by Leonhard Euler (leading mathematician at the time).
- 17xx** Daniel Bernoulli advocates use of infinite (as above) sums of sin and cos.
- 18xx** Jean Baptiste Joseph Fourier used these infinite series to study heat flow. Developed theory.

## Fourier Series: First Observations

For each positive integer  $n$ , the set of functions  $\{\Phi_0, \Phi_1, \dots, \Phi_{2n-1}\}$ , where

$$\begin{cases} \Phi_0(x) = \frac{1}{2} \\ \Phi_k(x) = \cos(kx), & k = 1, \dots, n \\ \Phi_{n+k}(x) = \sin(kx), & k = 1, \dots, n-1 \end{cases}$$

is an orthogonal set on the interval  $[-\pi, \pi]$  with respect to the weight function  $w(x) = 1$ .

## Orthogonality

Orthogonality follows from the fact that integrals over  $[-\pi, \pi]$  of  $\cos(kx)$  and  $\sin(kx)$  are zero, and products can be rewritten as sums:

$$\begin{cases} \sin \theta_1 \sin \theta_2 = \frac{\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)}{2} \\ \cos \theta_1 \cos \theta_2 = \frac{\cos(\theta_1 - \theta_2) + \cos(\theta_1 + \theta_2)}{2} \\ \sin \theta_1 \cos \theta_2 = \frac{\sin(\theta_1 - \theta_2) + \sin(\theta_1 + \theta_2)}{2} \end{cases}$$

Let  $\mathcal{T}_n$  be the set of all linear combinations of the functions  $\{\Phi_0, \Phi_1, \dots, \Phi_{2n-1}\}$ ; this is the set of trigonometric polynomials of degree  $\leq n$ .

For  $f \in C[-\pi, \pi]$ , we seek the *continuous least squares approximation* by functions in  $\mathcal{T}_n$  of the form

$$S_n(x) = \frac{a_0}{2} + a_n \cos(nx) + \sum_{k=1}^{n-1} (a_k \cos(kx) + b_k \sin(kx)),$$

where, thanks to orthogonality

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx.$$

**Definition:** — The limit

$$S(x) = \lim_{n \rightarrow \infty} S_n(x)$$

is called the **Fourier Series** of  $f$ .

First we note that  $f(x)$  and  $\cos(kx)$  are even functions on  $[-\pi, \pi]$  and  $\sin(kx)$  are odd functions on  $[-\pi, \pi]$ . Hence,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx = \frac{2}{\pi} \int_0^{\pi} x dx = \pi.$$

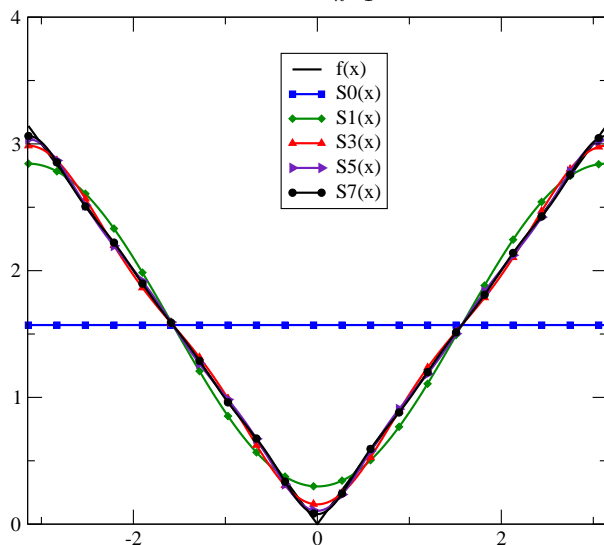
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(kx) dx = \frac{2}{\pi} \int_0^{\pi} x \cos(kx) dx$$

$$= \frac{2}{\pi} \underbrace{x \frac{\sin(kx)}{k}}_0 \Big|_0^{\pi} - \frac{2}{k\pi} \int_0^{\pi} 1 \cdot \sin(kx) dx$$

$$= \frac{2}{\pi k^2} [\cos(k\pi) - \cos(0)] = \frac{2}{\pi k^2} [(-1)^k - 1].$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{|x| \sin(kx)}_{\text{even} \times \text{odd} = \text{odd}} dx = 0.$$

We can write down  $S_n(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{k=1}^n \frac{(-1)^k - 1}{k^2} \cos(kx)$



The discrete Fourier transform, a.k.a. the finite Fourier transform, is a transform on samples of a function.

It, and its “cousins,” are the most widely used mathematical transforms; applications include:

- Signal Processing
  - Image Processing
  - Audio Processing
- Data compression
- A tool for partial differential equations
- etc...

Suppose we have  $2m$  data points,  $(x_j, f_j)$ , where

$$x_j = -\pi + \frac{j\pi}{m}, \text{ and } f_j = f(x_j), \quad j = 0, 1, \dots, 2m - 1.$$

The discrete least squares fit of a trigonometric polynomial  $S_n(x) \in \mathcal{T}_n$  minimizes

$$E(S_n) = \sum_{j=0}^{2m-1} [S_n(x_j) - f_j]^2.$$

We know that the basis functions

$$\begin{cases} \Phi_0(x) = \frac{1}{2} \\ \Phi_k(x) = \cos(kx), \quad k = 1, \dots, n \\ \Phi_{n+k}(x) = \sin(kx), \quad k = 1, \dots, n-1 \end{cases}$$

are orthogonal with respect to *integration* over the interval.

**The Big Question:** Are they orthogonal in the discrete case? Is the following true:

$$\sum_{j=0}^{2m-1} \Phi_k(x_j)\Phi_l(x_j) = \alpha_k \delta_{k,l} \quad ???$$

Orthogonality of the Basis Functions! (A Lemma)...

**Lemma:** — If the integer  $r$  is not a multiple of  $2m$ , then

$$\sum_{j=0}^{2m-1} \cos(rx_j) = \sum_{j=0}^{2m-1} \sin(rx_j) = 0.$$

Moreover, if  $r$  is not a multiple of  $m$ , then

$$\sum_{j=0}^{2m-1} [\cos(rx_j)]^2 = \sum_{j=0}^{2m-1} [\sin(rx_j)]^2 = m.$$

Proof of Lemma

Recalling long-forgotten (or quite possible never seen) facts from **Complex Analysis — Euler's Formula:**

$$e^{i\theta} = \cos(\theta) + i \sin(\theta).$$

Thus,

$$\sum_{j=0}^{2m-1} \cos(rx_j) + i \sum_{j=0}^{2m-1} \sin(rx_j) = \sum_{j=0}^{2m-1} [\cos(rx_j) + i \sin(rx_j)] = \sum_{j=0}^{2m-1} e^{irx_j}.$$

Since

$$e^{irx_j} = e^{ir(-\pi + j\pi/m)} = e^{-ir\pi} e^{irj\pi/m},$$

we get

$$\sum_{j=0}^{2m-1} \cos(rx_j) + i \sum_{j=0}^{2m-1} \sin(rx_j) = e^{-ir\pi} \sum_{j=0}^{2m-1} e^{irj\pi/m}.$$

Since  $\sum_{j=0}^{2m-1} e^{irj\pi/m}$  is a **geometric series** with first term 1, and ratio  $e^{ir\pi/m} \neq 1$ , we get

$$\sum_{j=0}^{2m-1} e^{irj\pi/m} = \frac{1 - (e^{ir\pi/m})^{2m}}{1 - e^{ir\pi/m}} = \frac{1 - e^{2ir\pi}}{1 - e^{ir\pi/m}}.$$

This is zero since

$$1 - e^{2ir\pi} = 1 - \cos(2r\pi) - i \sin(2r\pi) = 1 - 1 - i \cdot 0 = 0.$$

This shows the first part of the lemma:

$$\sum_{j=0}^{2m-1} \cos(rx_j) = \sum_{j=0}^{2m-1} \sin(rx_j) = 0.$$

If  $r$  is not a multiple of  $m$ , then

$$\sum_{j=0}^{2m-1} [\cos(rx_j)]^2 = \sum_{j=0}^{2m-1} \frac{1 + \cos(2rx_j)}{2} = \sum_{j=0}^{2m-1} \frac{1}{2} = m.$$

Similarly (use  $\cos^2 \theta + \sin^2 \theta = 1$ )

$$\sum_{j=0}^{2m-1} [\sin(rx_j)]^2 = m.$$

This proves the second part of the lemma.

We are now ready to show that the basis functions are orthogonal.

### Showing Orthogonality of the Basis Functions

Recall

$$\begin{cases} \sin \theta_1 \sin \theta_2 = \frac{\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)}{2} \\ \cos \theta_1 \cos \theta_2 = \frac{\cos(\theta_1 - \theta_2) + \cos(\theta_1 + \theta_2)}{2} \\ \sin \theta_1 \cos \theta_2 = \frac{\sin(\theta_1 - \theta_2) + \sin(\theta_1 + \theta_2)}{2}. \end{cases}$$

Thus for any pair  $k \neq l$

$$\sum_{j=0}^{2m-1} \Phi_k(x_j) \Phi_l(x_j)$$

is a zero-sum of sin or cos, and when  $k = l$ , the sum is  $m$ .

### Finally: The Trigonometric Least Squares Solution

Using

**[1]** Our standard framework for deriving the least squares solution — set the partial derivatives with respect to all parameters equal to zero.

**[2]** The orthogonality of the basis functions.

We find the coefficients in the summation

$$S_n(x) = \frac{a_0}{2} + a_n \cos(nx) + \sum_{k=1}^{n-1} (a_k \cos(kx) + b_k \sin(kx)) :$$

$$a_k = \frac{1}{m} \sum_{j=0}^{2m-1} f_j \cos(kx_j), \quad b_k = \frac{1}{m} \sum_{j=0}^{2m-1} f_j \sin(kx_j).$$

Let  $f(x) = x^3 - 2x^2 + x + 1/(x - 4)$  for  $x \in [-\pi, \pi]$ .

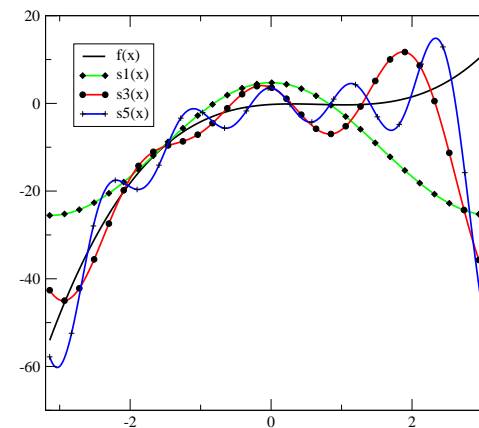
Let  $x_j = -\pi + j\pi/5$ ,  $j = 0, 1, \dots, 9$ , i.e.

$j$	$x_j$	$f_j$
0	-3.14159	-54.02710
1	-2.51327	-31.17511
2	-1.88495	-15.85835
3	-1.25663	-6.58954
4	-0.62831	-1.88199
5	0	-0.25
6	0.62831	-0.20978
7	1.25663	-0.28175
8	1.88495	1.00339
9	2.51327	5.08277

We get the following coefficients:

$$a_0 = -20.837, \quad a_1 = 15.1322, \quad a_2 = -9.0819, \quad a_3 = 7.9803$$

$$b_1 = 8.8661, \quad b_2 = -7.8193, \quad b_3 = 4.4910.$$



$S_1(x), S_3(x), S_5(x)$

Notes:

- [1] The approximation get better as  $n \rightarrow \infty$ .
- [2] Since all the  $S_n(x)$  are  $2\pi$ -periodic, we will always have a problem when  $f(-\pi) \neq f(\pi)$ . [Fix: Periodic extension.] On the following two slides we see the performance for a  $2\pi$ -periodic  $f$ .
- [3] It seems like we need  $\mathcal{O}(m^2)$  operations to compute  $\vec{a}$  and  $\vec{b}$  —  $m$  sums, with  $m$  additions and multiplications. There is however a fast  $\mathcal{O}(m \log_2(m))$  algorithm that finds these coefficients. We will talk about this **Fast Fourier Transform** next time.

Let  $f(x) = 2x^2 + \cos(3x) + \sin(2x)$   $x \in [-\pi, \pi]$ .

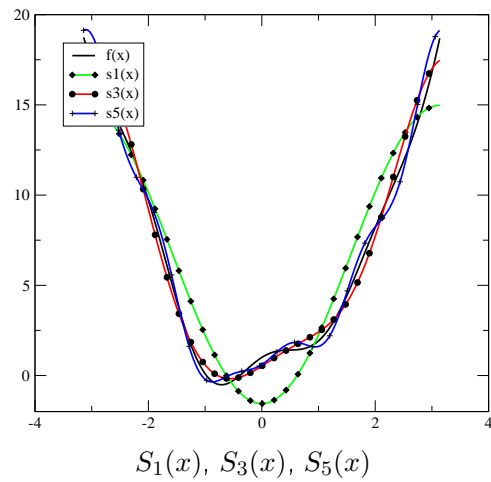
Let  $x_j = -\pi + j\pi/5$ ,  $j = 0, 1, \dots, 9$ , i.e.

$j$	$x_j$	$f_j$
0	-3.14159	18.7392
1	-2.51327	13.8932
2	-1.88495	8.5029
3	-1.25663	1.7615
4	-0.62831	-0.4705
5	0	1.0000
6	0.62831	1.4316
7	1.25663	2.9370
8	1.88495	7.3273
9	2.51327	11.9911

We get the following coefficients:

$$a_0 = -8.2685, \quad a_1 = 2.2853, \quad a_2 = -0.2064, \quad a_3 = 0.8729$$

$$b_1 = 0, \quad b_2 = 1, \quad b_3 = 0.$$



$S_1(x), S_3(x), S_5(x)$