I, ______________________________(your name), pledge that this exam is completely my own work, and that I did not take, borrow or steal work from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my work. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

1. The table below presents the population (in millions) for Japan.

Year	Population
1950	83.81
1960	94.09
1970	104.34
1980	116.81
1990	123.54
2000	126.70

a. Find the growth rate for each decade with the data above. Associate each growth rate with the earlier of the two census dates. Determine the average (mean) growth rate, r, from the data above. Associate t with the earlier of the dates in the growth ratio, and find the best straight line

$$k(t) = a + bt$$

through the growth data. Graph the constant function r, $k(t)$, and the data as a function of t over the period of the census data. Obtain at least 4 significant figures for a and b.

b. The Discrete Malthusian growth model is given by

$$P_{n+1} = (1 + r)P_n.$$

where r is computed in Part a, and P_0 is the population in 1950. Write the general solution to this model, where n is in decades. Use the model to predict the population in 2020 and 2050.

c. The revised growth model is given by

$$P_{n+1} = (1 + k(t_n))P_n.$$

where $k(t_n)$ is computed in Part a. $t_n = 1950 + 10n$, and P_0 is again the population in 1950. Simulate this nonautonomous discrete dynamical model from 1950 to 2050. Use the model to predict the population in 2020 and 2050. The growth rate of the nonautonomous dynamical model goes to zero during this century for Japan. At this time, this model predicts that the population will reach its maximum and start declining. Use the growth rate $k(t)$ to find when this model predicts a maximum population, then estimate what that maximum population will be.

d. Use the data above to find the best discrete logistic growth model fit for the population of Japan. (Again use P_0 as the population in 1950.) Add the graph of this model to your previous graph of the Malthusian growth models for the time period 1950 to 2050. What does this model predict for the population of Japan in 2050? From the sum of square errors, which model matches the data best? Find all equilibria of this model and discuss the stability of these equilibria (include the values of the derivatives at the equilibria). What does this model predict will happen over a long period of time for Japan's population?

e. Briefly discuss the strengths and weaknesses of each of these models. Which model does the best job of predicting the population into the future and for how long?

2. A. C. Crombie studied *Rhizopertha dominica*, the American wheat weevil, with an almost constant nutrient supply (maintained 10 g of cracked wheat weekly). These conditions match the assumptions of the discrete logistic model. The data below show the adult population of *Rhizopertha* from Crombie's study (with some minor modifications to fill in uncollected data and an initial shift of one week).

Week	Adults	Week	Adults
0	2	18	302
2	2	20	330
4	3	22	316
6	65	24	333
8	119	26	350
10	130	28	332
12	175	30	333
14	205	32	335
16	261	34	330

a. The discrete logistic growth model for the adult population P_n can be written

$$P_{n+1} = f(P_n) = rP_n - mP_n^2,$$

where the constants r and m must be determined from the data. In this part of the problem, you want to find the logistic growth updating function by graphing P_{n+1} vs. P_n, which you can do by entering the adult population data from times 2–34 for P_{n+1} and times 0–32 for P_n. Find the appropriate constants r and m by fitting the best quadratic (of the appropriate form) to these data. Graph both $f(P)$ and the data along with the identity map, $P_{n+1} = P_n$.

b. Find the equilibria for this model. Write the derivative of the updating function. Discuss the behavior of the model near its equilibria. Use the model found above to simulate the data. Find the best fitting model to the data by using the model found in Part a and adjusting the initial condition, P_0, to give the least squares best fit to the data. Write both the sum of square errors and P_0. Graph this simulation and the data (adult population vs. time). Discuss how well your simulation matches the data in the table. What do you predict will happen to the adult American wheat weevil population for large times (assuming experimental conditions continue)?

c. Another common population model is Hassell's model, which is given by

$$P_{n+1} = H(P_n) = \frac{aP_n}{(1+bP_n)^c},$$

where a, b, and c are constants to be determined. Suppose that it is found that for this type of beetle, the best value of $c = 2.4$. Find the least squares best fit of the Hassell's updating function to the given data by varying a and b. Give the values of a and b along with the sum of square errors for this fit. Once again graph both $H(P)$ and the data along with the identity map, $P_{n+1} = P_n$. How does this updating function compare to the one given in Part a?

d. Find the equilibria for Hassell's model. Write the derivative of this updating function. Discuss the behavior of this model near its equilibria. Use Hassell's model to simulate the data. Find the best fitting model to the data by using the model found in Part c and adjusting the initial condition, P_0, to give the least squares best fit to the data. Write both the sum of square errors and P_0. Graph this simulation and the data (adult population vs. time). Discuss how well your simulation matches the data in the table. Discuss the similarities and differences that you observe between these models and how well they work for this experimental situation.

3. It is known that certain species of birds need a critical population of birds before they can form a sustaining colony. This type of behavior gives rise to the Allee effect, and can be modeled by the differential equation:

$$\frac{dP}{dt} = P\left(r - a(P-b)^2\right) = F(P),$$

where P is the population of the birds, t is in years, and $F(P)$ is the growth of the population.

Suppose that a series of surveys finds that the annual growth of the population of this species of bird given its current population has the values in the table below.

Population P	Growth $F(P)$	Population P	Growth $F(P)$
100	-2	1000	83
400	16	1200	96
500	22	1500	98
800	59	1800	61

a. Use the data on the annual growth of this species of bird to fit the parameters r, a, and b in the Allee model given above.

b. Find all equilibria and determine the stability of the equilibria. Draw a phase portrait of this model, then briefly describe what you expect will happen to this species of bird for various population levels.

4. One of the common problems with drugs is that they become metabolized into another active or toxic substance. This complicates how one determines the correct dosage for a safe, but therapeutic treatment. Consider the drug prozac, fluoxetine, which is a selective serotonin reuptake inhibitor (SSRI) used for treatment of depression and other mood disorders. It happens that fluoxetine is metabolized quite rapidly with a half-life in the blood stream of only 1.5 days. A longer lasting, but not quite as potent, metabolite is norfluoxetine. A mathematical model for this type of drug interaction is given by the following system of differential equations:

$$\frac{dF}{dt} = -k_1 F,$$

$$\frac{dN}{dt} = a_1F - k_2N,$$

where $F(t)$ is the concentration of fluoxetine in the blood and $N(t)$ is the concentration of norfluoxetine in the blood. Suppose that an initial dose of fluoxetine is rapidly absorbed into the blood and $F(0) = 21$ ng/ml. Also, assume that that there is no norfluoxetine in the blood initially, so $N(0) = 0$ ng/ml. Careful monitoring shows that the norfluoxetine concentration peaks at $t_{max} = 4.5$ days with $N(t_{max}) = 16.0$ ng/ml.

Briefly explain what each of the terms in the model is describing in this drug treatment regimen. Use the information above to solve the system of differential equations and find the kinetic constants, k_1, k_2, and a_1. Determine the equilibria and give the eigenvalues and eigenvectors for this system. Graph the time series solutions for $F(t)$ and $N(t)$ for $t \in [0, 20]$. Also, produce a phase portrait of this system (with F on the horizontal axis). Show the specific solution in your phase portrait that satisfies the initial conditions for the time series solution.

5. G. F. Gause in his book *The Struggle for Existence* studied a number of competition and predator-prey systems in the lab. One predator-prey system that he studied was the interaction between the predator *Didinium nasutum* and its prey *Paramecium caudatum.* Below is a table for one of his experiments.

Day t	*P. caud.* P	*D. nasu.* N	Day t	*P. caud.* P	*D. nasu.* N
0	1	1	9	3	13
1	5	1	10	1	6
2	8	2	11	7	2
3	17	2	12	16	1
4	24	5	13	27	4
5	56	8	14	18	22
6	26	31	15	3	36
7	9	26	16	2	15
8	6	15	17	1	8

a. Use the material from the lynx and hare study in class to find the best fitting parameters to the predator-prey model given by:

$$\begin{aligned} \dot{P} &= a_1P - a_2PN, \\ \dot{N} &= -b1N + b_2PN, \end{aligned}$$

where P is the population of *Paramecium caudatum* and N is the population of *Didinium nasutum.* Give your best fitting parameters $P(0)$, $N(0)$, a_1, a_2, b_1, and b_2 along with the sum of square errors to the data. (Hint: This problem converges poorly for the wrong set of parameters, so you may want to either adjust a few parameters by hand until you are closer to matching the data or fix a few parameters and adjust only 2-4 others to get closer to the correct answer.)

b. Find all equilibria for this model, then discuss the stability of these equilibria, giving the eigenvalues. Characterize each of the equilibria (*e.g.*, stable node, saddle node, unstable spiral).

Is this model structurally stable? Produce a graph with the time evolution of both populations and another graph showing the phase portrait of the two populations (including arrows to show the direction of the solution). Be sure to include the data on your graph.

c. We also examined an alternative predator-prey model, where the model includes an intraspecies competition term for the prey species. This model is given by the system of differential equations

$$\begin{aligned} \dot{P} &= a_1P - a_2PN - a_3P^2, \\ \dot{N} &= -b1N + b_2PN. \end{aligned}$$

Repeat the process that you did in Parts a and b to find the new best set of parameters and its sum of square errors to the data. Find all equilibria with their eigenvalues, and characterize these equilibria. Is this model structurally stable? Once again, produce a graph with the time evolution of both populations and another graph showing the phase portrait of the two populations (including arrows to show the direction of the solution). Be sure to include the data on your graph. Briefly discuss which model appears to be better and why.

6. A common model used by fisheries is Ricker's model given by

$$\frac{dN}{dt} = N(re^{-\beta N} - 1),$$

where r and β are positive parameters determined by the ecosystem.

a. Find the equilibria for this model and determine the stability of these equilibria.

b. Let $r = 5$ and $\beta = 0.01$. Consider the two Ricker's models with fishing given by:

$$\begin{aligned} \frac{dN}{dt} &= N(re^{-\beta N} - 1) - h_1, \\ \frac{dN}{dt} &= N(re^{-\beta N} - 1) - h_2N, \end{aligned}$$

where $h_1 \geq 0$ is a constant level of fishing and $h_2 \geq 0$ is a level of fishing proportional to the population of fish. Find the maximum level of harvesting using each of these methods of fishing. Also, create a bifurcation diagram for each of these fishing models using either h_1 or h_2 as your bifurcation parameter. Be sure to state what type of bifurcation occurs in each case.

7. We studied the logistic growth equation, which has a stable carrying capacity. However, it has been argued that the density dependence factor could be delayed due to maturation times. Consider the delayed logistic growth model given by:

$$\frac{dP(t)}{dt} = rP(t)\left(1 - \frac{P(t-T)}{M}\right),$$

where r and M are positive parameters as before for the logistic growth model, and T is the delay for maturation.

a. As with the classic logistic growth model, the equilibria are $P_e = 0$ and M. Find the linearized model about $P_e = 0$ and determine its stability.

b. Linearize this model about the equilibrium $P_e = M$ and find the characteristic equation for this linearization. Show that a Hopf bifurcation occurs and determine where this happens.

c. Let $r = 0.5$ and $M = 500$. Simulate this model for $T = 2$ and $T = 4$, using constant initial data $P(t) = 100$ for $t \in [-T, 0]$.

8. a. Consider an animal that lives five years and reproduces annually. Animals that are 0-1 years old don't reproduce and only 35% ($s_1 = 0.35$) of them survive to the next year. Animals that are 1-2 years old are inexperienced and produce on average $b_2 = 0.7$ offspring and 60% ($s_2 = 0.6$) of them survive to the next year. Animals 2-3 years old produce on average $b_3 = 1.6$ offspring and 75% ($s_3 = 0.75$) of them survive to the next year. The peak age is animals 3-4 years old, who produce $b_4 = 2.5$ offspring and 70% ($s_4 = 0.70$) of them survive to the next year. Finally, animals 4-5 years old produce $b_5 = 1.8$ offspring. Create a model using a Leslie matrix, L, of the form:

$$P_{n+1} = LP_n.$$

Find the steady-state percentage of each age group. Determine how long it takes for this population to double after it has reached its steady-state distribution.

b. Assume that a fraction of 3-5 year olds are harvested. That is, the survival rates s_3 and s_4 are reduced. If the survival rates are reduced by a fraction α, so that the survival rate of 2-3 year olds is $0.75\,\alpha$ and the survival rate of 3-4 year olds is $0.7\,\alpha$. Determine the value of α that leaves the population at a constant value. For this value of α (to at least 4 significant figures), if there are 350 mature (4-5 year olds), then determine the total population and number in each population age group. How many animals are harvested annually under these conditions?