Math 342A — Take Home 2 — 11/12/04

1. a. From MAPLE, $\operatorname{rank}(A)=3$, $\det(A)=-38$, $\operatorname{rank}(B)=2$, $\det(B)$ undefined, $\operatorname{rank}(C)=3$, $\det(C)=27$

b. $AB=\begin{pmatrix}-2 & 7\\ 13 & 6\\ 12 & 6\end{pmatrix}$, $(AB)^TC=\begin{pmatrix}-115 & 4 & 39\\ -40 & 34 & 25\end{pmatrix}$, BA undefined, $2A-B$ undefined, $3A+C=\begin{pmatrix}0 & -1 & 7\\ 9 & 12 & 3\\ 6 & -6 & 15\end{pmatrix}$

c. $\det(A-\lambda I)=-(\lambda^3-6\lambda^2+3\lambda+38)=0$, e.v. $\lambda_1=-2$, e.f. $\xi_1=\begin{bmatrix}-2\\ 1\\ 1\end{bmatrix}$, $\lambda_2=4+i\sqrt{3}$, $\xi_2=\begin{bmatrix}4\\ 7-5i\sqrt{3}\\ 10+2i\sqrt{3}\end{bmatrix}$, $\lambda_3=\overline{\lambda}_2$, $\xi_3=\overline{\xi}_2$

All e.v.'s have multiplicity 1.

$\det|C-\lambda I|=(3-\lambda)^3=0 \Rightarrow \lambda_1=3$, $\xi_1=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}, \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$ algebraic multiply $=3$, geom. mult. $=2$

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2. a. $\det(Q)=-6a^2$. $\begin{bmatrix}4+a & -2-a & 0 & -4\\ 4 & -2 & 0 & -4\\ 0 & 3-a & 3 & -3+a\\ 4 & -2-a & 0 & -4+a\end{bmatrix} \xrightarrow[\text{Elim}]{\text{Gauss}} \begin{bmatrix}4 & -2 & 0 & -4\\ 0 & -a & 0 & a\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & a/2\end{bmatrix}$ $\operatorname{rank}(Q)=4$ if $a\neq 0$, $\operatorname{rank}(Q)=2$ if $a=0$.

From Maple, e.v. $\lambda_1=3$, $\xi_1=\begin{pmatrix}0\\ 0\\ 1\\ 0\end{pmatrix}$, $\lambda_2=-2$, $\xi_2=\begin{pmatrix}1\\ 1\\ 0\\ 1\end{pmatrix}$, $\lambda_3=a$, $\xi_3=\begin{pmatrix}-\frac{1}{2}-\frac{a}{4}\\ -1\\ 1\\ 0\end{pmatrix}, \begin{pmatrix}\frac{3}{2}+\frac{a}{4}\\ 1\\ 0\\ 1\end{pmatrix}$

For $a\neq 3$ or -2, alg. + geo. mult. λ_1 + λ_2 is 1, while alg + geo mult. λ_3 is 2.

For $a=3$, λ_2 has alg. + geo. mult. 1, while $\lambda_1=\lambda_3$ has alg + geo. mult of 3.

For $a=-2$, λ_1 has alg. + geo. mult. 1, while $\lambda_2=\lambda_3$ has alg mult 3 and geo. mult. 2.

b. Non-trivial solns iff $a=0$. From part a, with $a=0$, Gauss elimination gives

$\left[\begin{array}{cccc:c}2 & -1 & 0 & -2 & 0\\ 0 & 0 & 3 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right] \Rightarrow$ $x_3=0$, $x_4=t$, $x_2=s$, $x_1=\frac{s}{2}+t$ Nontrivial soln $\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}=\bar{s}\begin{pmatrix}1\\ 2\\ 0\\ 0\end{pmatrix}+t\begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix}$

If $a=5$, then $\vec{x}=\vec{0}$.

c. If $a=-2$, Q is not diagonalizable as there are only 3 linearly independent e.f.

For $a\neq -2$, take $S=\begin{bmatrix}0 & 1 & -\frac{1}{2}-\frac{a}{4} & \frac{3}{2}+\frac{a}{4}\\ 0 & 1 & -1 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}$, then $S^{-1}QS=\begin{bmatrix}3 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & a & 0\\ 0 & 0 & 0 & a\end{bmatrix}$

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3. $\left[\begin{array}{ccccc:c}1 & 1 & 1 & 1 & 1 & 4\\ 2 & 1 & 0 & 0 & 3 & 2\\ -2 & 0 & 2 & 1 & -4 & 0\\ 0 & 1 & 2 & 1 & -1 & 2\end{array}\right] \xrightarrow[R_3+2R_1]{R_2-2R_1} \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 4\\ 0 & -1 & -2 & -2 & 1 & -6\\ 0 & 2 & 4 & 3 & -2 & 8\\ 0 & 1 & 2 & 1 & -1 & 2\end{bmatrix} \xrightarrow[\substack{R_3+2R_2\\ R_4+R_2}]{\substack{-R_2\\ R_1+R_2}} \begin{bmatrix}1 & 0 & -1 & -1 & 2 & -2\\ 0 & 1 & 2 & 2 & -1 & 6\\ 0 & 0 & 0 & -1 & 0 & -4\\ 0 & 0 & 0 & -1 & 0 & -4\end{bmatrix} \xrightarrow[\substack{R_2+2R_3\\ R_1-R_3}]{\substack{-R_3\\ R_4-R_3}} \left[\begin{array}{ccccc:c}1 & 0 & -1 & 0 & 2 & 2\\ 0 & 1 & 2 & 0 & -1 & -2\\ 0 & 0 & 0 & 1 & 0 & 4\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right]$

$x_1=x_3-2x_5+2$
$x_2=-2x_3+x_5-2$
$x_4=4$
$x_3=\alpha$ $x_5=\beta$

$\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}=\alpha\begin{bmatrix}1\\ -2\\ 1\\ 0\\ 0\end{bmatrix}+\beta\begin{bmatrix}-2\\ 1\\ 0\\ 0\\ 1\end{bmatrix}+\begin{bmatrix}2\\ -2\\ 0\\ 4\\ 0\end{bmatrix}$

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4. a. $\begin{vmatrix}a & 0 & 0\\ 0 & 0 & -c\\ b & c & 0\end{vmatrix}=ac^2$, $\begin{vmatrix}a-\lambda & 0 & 0\\ 0 & -\lambda & -c\\ b & c & -\lambda\end{vmatrix}=-(\lambda-a)(\lambda^2+c^2)=0 \Rightarrow \lambda=a, \lambda=\pm ic$, For $\lambda_1=a$, $\begin{pmatrix}0 & 0 & 0\\ 0 & -a & -c\\ b & c & -a\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$

$x_1=\begin{pmatrix}a^2+c^2\\ -bc\\ ab\end{pmatrix}$, For $\lambda_2=ic$, $\begin{pmatrix}a-ic & 0 & 0\\ 0 & -ic & -c\\ b & c & -ic\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$ $\xi_1=0$ $x_2=\begin{pmatrix}0\\ i\\ 1\end{pmatrix}$, For $\lambda_3=-ic$, $x_3=\begin{pmatrix}0\\ -i\\ 1\end{pmatrix}$

b. This is a vector space, dimension $=3$. Basis, $\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}$

Take-Home 2 - (cont)

⑤ a. $\left[\begin{array}{ccc:c} 1 & 1 & 0 & 3 \\ 2 & -1 & -3 & 0 \\ 6 & -3 & -\alpha^2 & \alpha+3 \end{array}\right] \xrightarrow[R_3-3R_2]{R_2-2R_1} \left[\begin{array}{ccc:c} 1 & 1 & 0 & 3 \\ 0 & -3 & -3 & -6 \\ 0 & 0 & 9-\alpha^2 & \alpha+3 \end{array}\right] \xrightarrow[R_1+\frac{R_2}{3}]{-\frac{R_2}{3}} \left[\begin{array}{ccc:c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 9-\alpha^2 & \alpha+3 \end{array}\right]$

Unique soln $\alpha \neq \pm 3$
No Soln if $\alpha = 3$
Infinitely many $\alpha = -3$

b. $\alpha = -3$ $\left[\begin{array}{ccc:c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = 1 + x_3$, $x_2 = -x_3 + 2$, $x_3 \cancel{=\alpha}$ arb. $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

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⑥ a. $\dim(V) = 3$. Basis $e_1 = x^3, e_2 = x^2, e_3 = 1$

b. W with $p(0) = 0 \Rightarrow a_0 = 0$ $\quad W = \{ p \mid p(x) = a_3x^3 + a_2x^2,\ x \in [0,1],\ a_i \in \mathbb{R} \}$ $\quad \dim(W) = 2$.

Basis $e_1 = x^3$, $e_2 = x^2$

c. $\langle p_3(x), a_3x^3 + a_2x^2 \rangle = \int_0^1 (a_3x^6 + a_2x^5)\,dx = \frac{a_3x^7}{7} + \frac{a_2x^6}{6}\Big|_0^1 = \frac{a_3}{7} + \frac{a_2}{6} = 0$. $\therefore$ any element of W of the form $\alpha(7x^3 - 6x^2)$ is orthogonal to $p_3(x) = x^3$.

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⑦ 4 modes $-i_1 + i_4 + i_5 = 0$, $-i_0 + i_3 - i_4 = 0$, $i_2 - i_3 - i_5 = 0$, $i_0 + i_1 - i_2 = 0$

3 loops $i_1R_1 + i_2R_2 + i_5R_5 = E_5$, $i_0R_0 - i_1R_1 - i_4R_4 = E_0$, $i_0R_0 + i_2R_2 + i_3R_3 = E_0$

$$\left[\begin{array}{cccccc:c} 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 10 & 20 & 0 & 0 & 20 & 100 \\ 15 & -10 & 0 & 0 & -30 & 0 & 50 \\ 15 & 0 & 20 & 5 & 0 & 0 & 50 \end{array}\right] \xrightarrow[\text{(MAPLE)}]{\text{rref}} \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 250/267 \\ 0 & 1 & 0 & 0 & 0 & 0 & 90/89 \\ 0 & 0 & 1 & 0 & 0 & 0 & 520/267 \\ 0 & 0 & 0 & 1 & 0 & 0 & -160/267 \\ 0 & 0 & 0 & 0 & 1 & 0 & -410/267 \\ 0 & 0 & 0 & 0 & 0 & 1 & 680/267 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, $i_0 = \frac{250}{267}$, $i_1 = \frac{90}{89}$, $i_2 = \frac{520}{267}$, $i_3 = -\frac{160}{267}$, $i_4 = -\frac{410}{267}$, $i_5 = \frac{680}{267}$