Math 342A — **Take Home 2** — **11/12/04**

1 a. From MAPLE, rank$(A) = 3$, $\det(A) = -38$, rank$(B) = 2$, $\det(B)$ undefined, rank$(C) = 3$
$\det(C) = 27$

b. $AB = \begin{pmatrix} -2 & 7 \\ 13 & 6 \\ 12 & 6 \end{pmatrix}$, $(AB)^T C = \begin{pmatrix} -115 & 4 & 39 \\ -40 & 34 & 25 \end{pmatrix}$, BA undefined, $2A - B$ undefined, $3A + C = \begin{pmatrix} 0 & -1 & 7 \\ 9 & 12 & 3 \\ 6 & -6 & 15 \end{pmatrix}$

c. $\det(A - \lambda I) = -(\lambda^3 - 6\lambda^2 + 3\lambda + 38) = 0$, e.v. $\lambda_1 = -2$, e.f. $\xi_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\lambda_2 = 4 + i\sqrt{3}$, $\xi_2 = \begin{bmatrix} 4 \\ 7 - 5i\sqrt{3} \\ 10 + 2i\sqrt{3} \end{bmatrix}$, $\lambda_3 = \overline{\lambda}_2$, $\xi_3 = \overline{\xi}_2$

All e.v.'s have multiplicity 1.
$\det|C - \lambda I| = (3 - \lambda)^3 = 0 \Rightarrow \lambda_1 = 3$, $\xi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ algebraic multiply $= 3$, geom. mult. $= 2$

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2 a. $\det(Q) = -6a^2$. $\begin{bmatrix} 4+a & -2-a & 0 & -4 \\ 4 & -2 & 0 & -4 \\ 0 & 3-a & 3 & -3+a \\ 4 & -2-a & 0 & -4+a \end{bmatrix} \xrightarrow[\text{Elim}]{\text{Gauss}} \begin{bmatrix} 4 & -2 & 0 & -4 \\ 0 & -a & 0 & a \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & a/2 \end{bmatrix}$ rank$(Q) = 4$ if $a \neq 0$
rank$(Q) = 2$ if $a = 0$.

From Maple, e.v. $\lambda_1 = 3$, $\xi_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. $\lambda_2 = -2$, $\xi_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\lambda_3 = a$, $\xi_3 = \begin{pmatrix} -\frac{1}{2} - \frac{a}{4} \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{3}{2} + \frac{a}{4} \\ 1 \\ 0 \\ 1 \end{pmatrix}$

For $a \neq 3$ or -2, alg. + geo. mult. λ_1 + λ_2 is 1, while alg + geo mult. λ_3 is 2.
For $a = 3$, λ_2 has alg. + geo. mult. 1, while $\lambda_1 = \lambda_3$ has alg + geo. mult of 3.
For $a = -2$, λ_1 has alg. + geo. mult. 1, while $\lambda_2 = \lambda_3$ has alg mult 3 and geo. mult. 2.

b. Non-trivial solns iff $a = 0$. From part a, with $a = 0$, Gauss elimination gives

$\left[\begin{array}{cccc|c} 2 & -1 & 0 & -2 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$ $x_3 = 0$, $x_2 = s$, $x_4 = t$, $x_1 = \frac{s}{2} + t$. Nontrivial soln $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

If $a = 5$, then $\vec{x} = \vec{0}$.

c. If $a = -2$, Q is not diagonalizable as there are only 3 linearly independent e.f.
For $a \neq -2$, take $S = \begin{bmatrix} 0 & 1 & -\frac{1}{2} - \frac{a}{4} & \frac{3}{2} + \frac{a}{4} \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, then $S^{-1} Q S = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$

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3 $\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 4 \\ 2 & 1 & 0 & 0 & 3 & 2 \\ -2 & 0 & 2 & 1 & -4 & 0 \\ 0 & 1 & 2 & 1 & -1 & 2 \end{array}\right] \xrightarrow[R_3 + 2R_1]{R_2 - 2R_1} \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 4 \\ 0 & -1 & -2 & -2 & 1 & -6 \\ 0 & 2 & 4 & 3 & -2 & 8 \\ 0 & 1 & 2 & 1 & -1 & 2 \end{array}\right] \xrightarrow[R_3 + 2R_2,\ R_4 + R_2]{-R_2,\ R_1 + R_2} \left[\begin{array}{cccccc} 1 & 0 & -1 & -1 & 2 & -2 \\ 0 & 1 & 2 & 2 & -1 & 6 \\ 0 & 0 & 0 & -1 & 0 & -4 \\ 0 & 0 & 0 & -1 & 0 & -4 \end{array}\right] \xrightarrow[R_2 + 2R_3,\ R_1 - R_3]{-R_3,\ R_4 - R_3} \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = x_3 - 2x_5 + 2$
$x_2 = -2x_3 + x_5 - 2$
$x_4 = 4$
$x_3 = \alpha$ $x_5 = \beta$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}$

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4 a. $\begin{vmatrix} a & 0 & 0 \\ 0 & 0 & -c \\ b & c & 0 \end{vmatrix} = ac^2$, $\begin{vmatrix} a-\lambda & 0 & 0 \\ 0 & -\lambda & -c \\ b & c & -\lambda \end{vmatrix} = -(\lambda - a)(\lambda^2 + c^2) = 0 \Rightarrow \lambda = a, \lambda = \pm ic$, For $\lambda_1 = a$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -a & -c \\ b & c & -a \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$x_1 = \begin{pmatrix} a^2 + c^2 \\ -bc \\ ab \end{pmatrix}$, For $\lambda_2 = ic$, $\begin{pmatrix} a - ic & 0 & 0 \\ 0 & -ic & -c \\ b & c & -ic \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\xi_1 = 0$ $x_2 = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$, For $\lambda_3 = -ic$, $x_3 = \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$

b. This is a vector space, dimension $= 3$. Basis, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

Take-Home 2 - (cont)

(5) a. $\left[\begin{array}{ccc:c} 1 & 1 & 0 & 3 \\ 2 & -1 & -3 & 0 \\ 6 & -3 & -\alpha^2 & \alpha+3 \end{array}\right] \xrightarrow[R_3-3R_2]{R_2-2R_1} \left[\begin{array}{ccc:c} 1 & 1 & 0 & 3 \\ 0 & -3 & -3 & -6 \\ 0 & 0 & 9-\alpha^2 & \alpha+3 \end{array}\right] \xrightarrow[R_1+\frac{R_2}{3}]{-\frac{R_2}{3}} \left[\begin{array}{ccc:c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 9-\alpha^2 & \alpha+3 \end{array}\right]$

Unique soln $\alpha \neq \pm 3$
No soln if $\alpha = 3$
Infinitely many $\alpha = -3$

b. $\alpha = -3$ $\left[\begin{array}{ccc:c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = 1 + x_3$, $x_2 = -x_3 + 2$, x_3 ~~=α~~ arb. $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

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(6) a. $\dim(V) = 3$. Basis $e_1 = x^3, e_2 = x^2, e_3 = 1$

b. W with $p(0) = 0 \Rightarrow a_0 = 0$ $W = \{ p \mid p(x) = a_3x^3 + a_2x^2,\ x \in [0,1],\ a_i \in \mathbb{R} \}$ $\dim(W) = 2$.

Basis $e_1 = x^3$, $e_2 = x^2$

c. $\langle p_3(x), a_3x^3 + a_2x^2 \rangle = \int_0^1 (a_3x^6 + a_2x^5)\,dx = \frac{a_3x^7}{7} + \frac{a_2x^6}{6}\Big|_0^1 = \frac{a_3}{7} + \frac{a_2}{6} = 0$. $\therefore$ any element of W of the form $\alpha(7x^3 - 6x^2)$ is orthogonal to $p_3(x) = x^3$.

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(7) 4 nodes $-i_1 + i_4 + i_5 = 0$, $-i_0 + i_3 - i_4 = 0$, $i_2 - i_3 - i_5 = 0$, $i_0 + i_1 - i_2 = 0$

3 loops $i_1R_1 + i_2R_2 + i_5R_5 = E_5$, $i_0R_0 - i_1R_1 - i_4R_4 = E_0$, $i_0R_0 + i_2R_2 + i_3R_3 = E_0$

$\left[\begin{array}{cccccc:c} 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 10 & 20 & 0 & 0 & 20 & 100 \\ 15 & -10 & 0 & 0 & -30 & 0 & 50 \\ 15 & 0 & 20 & 5 & 0 & 0 & 50 \end{array}\right] \xrightarrow[\text{(MAPLE)}]{\text{rref}} \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 250/267 \\ 0 & 1 & 0 & 0 & 0 & 0 & 90/89 \\ 0 & 0 & 1 & 0 & 0 & 0 & 520/267 \\ 0 & 0 & 0 & 1 & 0 & 0 & -160/267 \\ 0 & 0 & 0 & 0 & 1 & 0 & -410/267 \\ 0 & 0 & 0 & 0 & 0 & 1 & 680/267 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Thus, $i_0 = \frac{250}{267}$, $i_1 = \frac{90}{89}$, $i_2 = \frac{520}{267}$, $i_3 = -\frac{160}{267}$, $i_4 = -\frac{410}{267}$, $i_5 = \frac{680}{267}$